An Econometric Analysis of Some Models for Constructed Binary Time Series

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Prevalence of constructed binary variables

In macro/finance many studies feature *constructed* binary variables (S_t)
S_t is constructed from some observed series y_t

1. Expansion($S_t = 1$)/Contractions ($S_t = 0$). Constructed from GDP, Unemployment, Industrial Production...
2. Bull ($S_t = 1$)/Bear($S_t = 0$) markets. Constructed from S&P, NASDAQ, ASX...
3. Financial Crisis and Vulerability ($S_t = 1$), no crisis, not vulnerable ($S_t = 0$). Constructed from ex rates, interest rates, bank failures...
4. Hot ($S_t = 1$) and cold ($S_t = 0$) markets. Constructed from IPO's, housing prices, commodity prices
5. Many test statistics e.g. measuring relations between variables at extremes - "co-exceedances" $S_t = 1(\, y_{1t} > c, x_{1t} > c)$

Some frequently asked questions

- Why construct binary states from continuous data?
  - Literatures seek to **focus attention** on particular events such as recessions, financial crises.
- Why construct two states rather than three or four states.
  - segmenting data into several parts has been tried many times and failed.
  - Makes it harder to focus attention on aspects of interest
  - Curse of dimensionality
  - Transitions between states more complex.
- Isn't there some loss of information involved in construction?
  - After construction the researcher has $S_1(y_1, ..., y_T), ..., S_T(y_1, ..., y_T)$ and $\{y_1, ..., y_T\}$. No new information and no information lost.
  - Organized differently. Can interact S_t with y_t and x_t.
  - Analogy with permanent and transitory components.
  - Does it produce insights?

Summarizing methods of construction

Three equations

$$\wedge_t = \lambda_1 \left(\mathcal{Y}_{t-k}^{t+k}, S_t, r \right) \tag{1}$$

$$\vee_t = \lambda_2 \left(\mathcal{Y}_{t-k}^{t+k}, S_t, r \right) \tag{2}$$

$$S_{t+1} = \varphi \left(\mathbb{S}_{t-r}^{t}, \wedge_t, \vee_t, r \right) \tag{3}$$

S_t is the current state, $\wedge_t$ and $\vee_t$ are binary time series representing peaks and troughs. $\mathcal{Y}_{t-k}^{t+h} = \{y_{t-k}, \cdots, y_{t+h}\}$. And r is an integer one less than the minimum phase duration. Note cannot both be a peak and trough at time t. So,

$$\wedge_t \times \vee_t = 0 \tag{4}$$

View $\wedge_t$ and $\vee_t$ as internal to (algorithm/expert/committee)

Two representations: Pure turning point rule and termination rule

Pure turning point rule $\lambda_1\left(\mathcal{Y}_{t-k}^{t+k}, S_t, r\right)$ does not depend on $\left(S_t, r\right)$
For example pure turning point version of a calculus rule is

$$\lambda_1\left(\mathcal{Y}_{t-k}^{t+k}\right) = \mathbf{1}\left(\Delta y_{t+1} < 0, \Delta y_{t+1} > 0\right) \qquad \text{(Peak)}$$

$$\lambda_2\left(\mathcal{Y}_{t-k}^{t+k}\right) = \mathbf{1}\left(\Delta y_{t+1} > 0, \Delta y_{t+1} < 0\right) \qquad \text{(Trough)}$$

Equivalent termination rule version with $r = 0$ is

$$\lambda_1\left(\mathcal{Y}_{t-k}^{t+k}, S_t, 0\right) = \mathbf{1}\left(\Delta y_{t+1} < 0\right) S_t \qquad \text{(Peak)}$$

$$\lambda_1\left(\mathcal{Y}_{t-k}^{t+k}, S_t, 0\right) = \mathbf{1}\left(\Delta y_{t+1} > 0\right)\left(1 - S_t\right) \qquad \text{(Trough)}$$

Where $\mathbf{1}\left(z\right) = 1$ if statement z is true and $\mathbf{1}\left(z\right) = 0$ otherwise.

Combining internal turning points to make business cycle states

No censoring (r=0)

$$\varphi\left(\mathbb{S}_{t-1}^{t}, \wedge_t, \vee_t, 0\right) = S_t + (1 - S_t) \vee_t - S_t \wedge_t \tag{5}$$

Min completed phase duration two periods

$$\varphi\left(\mathbb{S}_{t-1}^{c,t}, \wedge_t, \vee_t, 1\right) = S_t^c + (1 - S_t^c)(1 - S_{t-1}^c) \vee_t - S_t^c S_{t-1}^c \wedge_t \tag{6}$$

General case min phase duration $r + 1$ periods

$$\varphi\left(\mathbb{S}_{t-1}^{c,t}, \wedge_t, \vee_t, r\right) = S_t^c + \prod_{i=0}^{r} (1 - S_{t-i}^c) \vee_t - \prod_{i=0}^{r} S_{t-i}^c \wedge_t \tag{7}$$

"Every process is (almost) Markov" (Meyn 2007, p538)

Natural to work with Markov representation

Because $\{\wedge_t, \vee_t\}$ are binary DGP for $\varphi\left(\mathbb{S}_{t-1}^{t}, \wedge_t, \vee_t, r\right)$ can (after some rearranging) be written as

$$
\begin{aligned}
\varphi\left(\mathbb{S}_{t-1}^{t}, \wedge_t, \vee_t, r\right) &= \varphi\left(\mathbb{S}_{t-r}^{t}, 0, 0, r\right) \\
&\quad + \left[\varphi\left(\mathbb{S}_{t-r}^{t}, 0, 1, r\right) - \varphi\left(\mathbb{S}_{t-r}^{t}, 0, 0, r\right)\right] \vee_t \\
&\quad + \left[\varphi\left(\mathbb{S}_{t-r}^{t}, 1, 0, r\right) - \varphi\left(\mathbb{S}_{t-r}^{t}, 0, 0, r\right)\right] \wedge_t
\end{aligned}
$$

Nature of representation depends on what econometrician observes

- Focus on case where $\wedge_t$, $\vee_t$ are not observed by econometrician but S_t^c is observed so condition on $\mathbb{S}_{t-q}^{c,t}$ where $q \geq r$.

$$E\left(S_{t+1}^{c} | \mathbb{S}_{t-q}^{c,t}\right) = E\left[\varphi\left(\mathbb{S}_{t-q}^{c,t}, \wedge_t, \vee_t\right) | \mathbb{S}_{t-q}^{c,t}\right]$$

Items of interest are $E\left(\wedge_t | \mathbb{S}_{t-q}^{c,t}\right)$ and $E\left(\vee_t | \mathbb{S}_{t-q}^{c,t}\right)$ but now S_{t+1}^{c} is clearly a markov process.

Example r=1, q=1, Phase censoring and MP(2)

Conditional expectation of peak is

$$E\left(\wedge_t|\mathbb{S}^{ct}_{t-1}\right) = \alpha_{00}\left(1-S^c_t\right)\left(1-S^c_{t-1}\right)+\alpha_{10}S^c_t\left(1-S^c_{t-1}\right) + \alpha_{01}\left(1-S^c_t\right)S^c_{t-1}+\alpha_{11}S^c_tS^c_{t-1}$$

Conditional expectation of trough is

$$E\left(\vee_t|\mathbb{S}^{ct}_{t-1}\right) = \beta_{00}\left(1-S^c_t\right)\left(1-S^c_{t-1}\right)+\beta_{10}S^c_t\left(1-S^c_{t-1}\right) + \beta_{01}\left(1-S^c_t\right)S^c_{t-1}+\beta_{11}S^c_tS^c_{t-1}$$

Thus

$$\begin{aligned} E\left(S^c_{t+1}|\mathbb{S}^{ct}_{t-1}\right) &= S^c_t+\beta_{00}\left(1-S^c_t\right)\left(1-S^c_{t-1}\right)-\alpha_{11}S^c_tS^c_{t-1} \\ &= \beta_{00}+\left(1-\beta_{00}\right)S^c_t-\beta_{00}S^c_{t-1}+\left(\beta_{00}-\alpha_{11}\right)S^c_tS^c_{t-1} \end{aligned}$$

Reduces to MP(1) if $\beta_{00}=\alpha_{11}=0$. That is obtain an MP(1) representation only if stay in initial state for ever. Points to problem with literature that estimates a static probit (which has a $MP(0)$ representation)

The problem with standard discrete choice models

The standard discrete choice model

$$E(S_{t+1}|\mathbb{S}_{t-1}^{t}, x_t) = F(\theta_0 + x_t'\beta + \theta_1 S_{t-1}^c + \theta_2 S_{t-2}^c + \theta_3 S_{t-1}^c S_{t-2}^c)$$

Where F() is cdf and $-\infty < \theta_i < \infty$. Probit model $F(\cdot) = \Phi(\cdot)$
Censoring means that

$$1 = E(S_{t+1}|S_t = 1, S_{t-1} = 0, x_t) = F(\theta_0 + x_t'\beta + \theta_1)$$

$$0 = E(S_{t+1}|S_t = 0, S_{t-1} = 1, x_t) = F(\theta_0 + x_t'\beta + \theta_2)$$

Requires $\theta_1 = \infty$ and $\theta_2 = -\infty$. Loglikelihood undefined!
Shows up as numerical problems when try and estimate $DDC(2)$.

Adding covariates

Incorporate covariates (x_t) into a Markov Process structure. Assume order 2.

$$S_t^c = \alpha(x_t) + \beta(x_t) S_{t-1}^c + \gamma(x) S_{t-2}^c + \delta(x) S_{t-1}^c S_{t-2}^c + \eta_t, \qquad (8)$$

$\alpha(x_t), \beta(x_t), \gamma(x_t), \delta(x_t)$are non-linear functions of x_t
Phase censoring restrictions require

$$\begin{aligned} \alpha(x_t) + \beta(x_t) &= 1 \\ \alpha(x_t) + \gamma(x_t) &= 0 \end{aligned}$$

$$\Longrightarrow \Delta S_t^c = \alpha(x_t)(1 - S_{t-1}^c - S_{t-2}^c) + \delta(x_t) S_{t-1}^c S_{t-2}^c + \eta_t$$

Could use splines etc for $\alpha(x_t), \delta(x_t)$
Instead will use kernel based estimators

Conditional expectations implied by the MP(2) with covariates

$$
\begin{aligned}
E(S_t^c|S_{t-1}^c &= 1, S_{t-2}^c = 1, x_t) = \alpha(x_t) + \beta(x_t) + \gamma(x_t) + \delta(x_t) \\
E(S_t^c|S_{t-1}^c &= 1, S_{t-2}^c = 0, x_t) = \alpha(x_t) + \beta(x_t) \\
E(S_t^c|S_{t-1}^c &= 0, S_{t-2}^c = 1, x_t) = \alpha(x_t) + \gamma(x_t) \\
E(S_t^c|S_{t-1}^c &= 0, S_{t-2}^c = 0, x_t) = \alpha(x_t),
\end{aligned}
$$

$$
E(S_t^c|x_t) = \sum_{j=0}^{1}\sum_{k=0}^{1} E(S_t^c|S_{t-1}^c = j, S_{t-2}^c = k, x_t)\Pr(S_{t-1}^c = j, S_{t-2}^c = k|x_t).
$$

Consequently, it is necessary to evaluate a number of conditional expectations.

Estimate the expected values and probabilities from kernel methods

$$\hat{E}(S_t^c | S_{t-1}^c = j, S_{t-2}^c = k, x) = \frac{\sum_{t \in I_{jk}} S_t^c K_x\left(\frac{x_t - x}{h}\right)}{\sum_{t \in I_{jk}} K_x\left(\frac{x_t - x}{h}\right)} \quad (9)$$

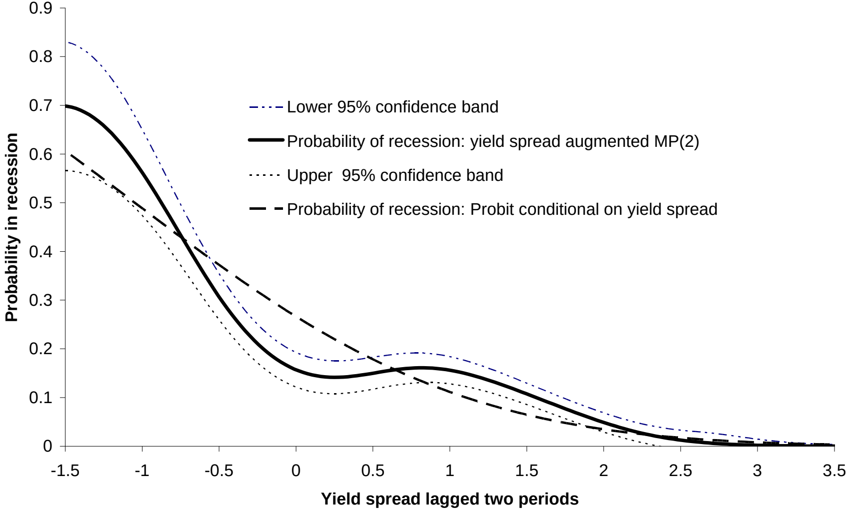
$$\Pr(S_{t-1}^c = j, S_{t-2}^c = k | x_t) = \frac{\sum_{t \in I_{jk}}^{T} K\left(\frac{x_t - x}{h}\right)}{\sum_{t=1}^{T} K\left(\frac{x_t - x}{h}\right)} \quad (10)$$

$I_{jk} = \{t \text{ s.t. } S_{t-1}^c = j, S_{t-2}^c = k\}$,
Can get asymptotic variance using standard methods. Li and Racine(2007 theorem 18.4)

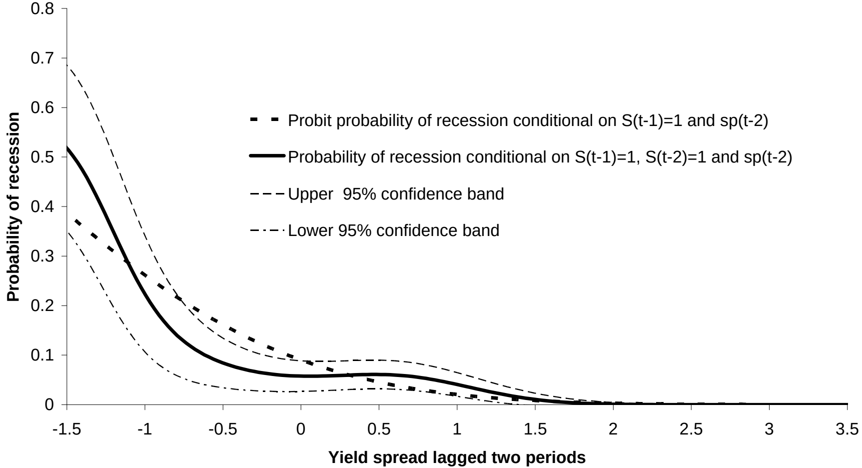
An Application to the Probability of Recessions Given the Yield Spread

sp_t is yield spread
S_t^c is *NBER* business cycle state ($S_t^c = 1$ in expansion), ($S_t^c = 0$ in contraction)
Using same data as Estrella and Mishkin (1998).
$x_t = sp_{t-2}$ in this application.
Use Gaussian kernel for x_t and indicator kernel for S_t. Later is based on Li and Racine (2007).

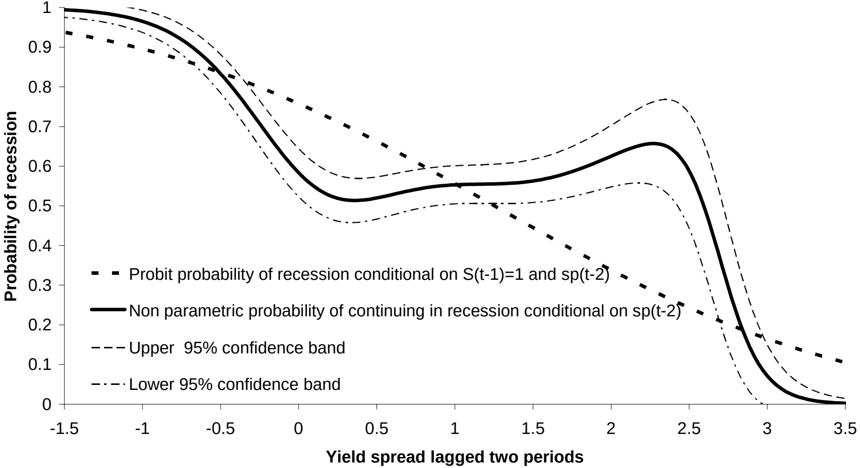
Probability of recession from MP(2) and Probit models conditional on the yield spread lagged two quarters



Probability end an expansion that has lasted two or more quarters conditional on the yield spread



Probability exit a recession conditional on yield spread



Take away points

1. Can formalize methods of construction into three equations
2. S_t has a Markov Process (MP) representation
   - MP(q) should be basis for all empirical work with constructed binary variables.
3. Method of construction places restrictions on MP
   - Order of MP >1.
4. Standard static (SDC) and dynamic discrete choice (DDC) models inappropriate
   - Likelihood is undefined (some parameters must be infinite)
   - Is possible to specify non standard dynamic discrete choice model for this data.
5. Feasible alternative via non-parametrics
   - Should always estimate this as a check on parametric models

Seeking help

I am interested in applying these methods to

- A range of other countries
- Study the effect of fiscal policy
- But,... I need suggestions for public data and comments on what I have done. Am visiting the following ECB, BIS, OECD to seek help.
    - IMF has a dataset but it is not yet public.