Forecasting Aggregated Time Series Variables
A Survey

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Issues of Interest

- Forecasting macroeconomic variables is the theme of the conference
- Macro variables are often aggregates
- Types of aggregation:
  - temporal aggregation over a number of periods
  - contemporaneous aggregation
    - over countries, e.g., EU or euro-area
    - over regions of a specific country, e.g., Bundesländer in Germany
    - over sectors of an economy

Overview

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VARMA model

Variables of interest: $y_t = (y_{1t}, \dots, y_{Kt})'$

$$y_t = A_1 y_{t-1} + \cdots + A_p y_{t-p} + u_t + M_1 u_{t-1} + \cdots + M_q u_{t-q}, \quad t \in \mathbb{Z}$$

or

$$A(L) y_t = M(L) u_t, \qquad t \in \mathbb{Z}$$

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with

- $A(L) = I_K - A_1 L - \cdots - A_p L^p$
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- $u_t \sim (0, \Sigma_u)$
- $\det A(z) \neq 0,\ |z| \leq 1$ **and** $\det M(z) \neq 0,\ |z| \leq 1$ **for** $z \in \mathbb{C}$

Alternative representations

Wold MA representation

$$y_t = \sum_{i=0}^{\infty} \Phi_i u_{t-i}$$

with $\Phi(L) = I_K + \sum_{i=1}^{\infty} \Phi_i L^i = A(L)^{-1} M(L)$

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Deterministic terms

$$y_t = \mu_t + x_t$$

Forecasting

Minimum MSE *h*-step forecast

$$y_{\tau+h|\tau} \equiv E(y_{\tau+h}|y_\tau, y_{\tau-1}, \dots)$$

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Forecasting

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MSE matrix

$$\Sigma_y(h) \equiv E[(y_{\tau+h} - y_{\tau+h|\tau})(y_{\tau+h} - y_{\tau+h|\tau})'] = \sum_{j=0}^{h-1} \Phi_j \Sigma_u \Phi_j'$$

$$\Sigma_y(h) \to_{h\to\infty} \Sigma_y \equiv E(y_t y_t') \text{ for stationary process}$$

Overview

Three predictors

Transformed process of interest: $z_t = Fy_t$

e.g., $z_t = \sum_{k=1}^{K} y_{kt} = [1, \ldots, 1]y_t$

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$$z_{\tau+h|\tau}$$

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Aggregate forecasts of individual components of disaggregated process

$$z^u_{\tau+h|\tau} \equiv F y^u_{\tau+h|\tau}$$

Properties of predictors

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- $\Sigma_z(h) \geq \Sigma_z^o(h)$
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  (Forecasting disaggregated process and aggregating forecasts is optimal)
- $\Sigma_z(h) \geq \Sigma_z^u(h)$ or $\Sigma_z(h) \leq \Sigma_z^u(h)$

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- $\Sigma_z(h) \geq \Sigma_z^u(h)$ or $\Sigma_z(h) \leq \Sigma_z^u(h)$
- For stationary processes:

  $$\Sigma_z(h), \Sigma_z^o(h), \Sigma_z^u(h) \to \Sigma_z \equiv E(z_t z_t') \quad \text{as} \quad h \to \infty$$

Equality of predictors

Tiao and Guttman (1980), Kohn (1982), Lütkepohl (1984) give results on equality of predictors

Examples

- $z^o_{\tau+1|\tau} = z^u_{\tau+1|\tau}$ if y_t consists of independent components
- $z^o_{\tau+1|\tau} = z_{\tau+1|\tau}$ if y_t consists of independent components generated by the same ARMA process and $F = [1, \dots, 1]$

Overview

Results for estimated processes

Forecast based on estimated parameters: $\hat{y}_{\tau+h|\tau}$

Forecast error

$$y_{\tau+h} - \hat{y}_{\tau+h|\tau} = (y_{\tau+h} - y_{\tau+h|\tau}) + (y_{\tau+h|\tau} - \hat{y}_{\tau+h|\tau})$$

Forecast MSE

$$\Sigma_{\hat{y}}(h) = \text{MSE}(y_{\tau+h|\tau}) + \text{MSE}(y_{\tau+h|\tau} - \hat{y}_{\tau+h|\tau})$$

$$= \Sigma_y(h) + E[(y_{\tau+h|\tau} - \hat{y}_{\tau+h|\tau})(y_{\tau+h|\tau} - \hat{y}_{\tau+h|\tau})']$$

$$\approx \Sigma_y(h) + \frac{1}{T}\Omega(h)$$

$$\text{where } \Omega(h) = E\left[\frac{\partial y_{\tau+h|\tau}}{\partial \theta'} \Sigma_{\tilde{\theta}} \frac{\partial y'_{\tau+h|\tau}}{\partial \theta}\right]$$

Yamamoto (1980), Baillie (1981), Lütkepohl (2005)

Implications for aggregated processes

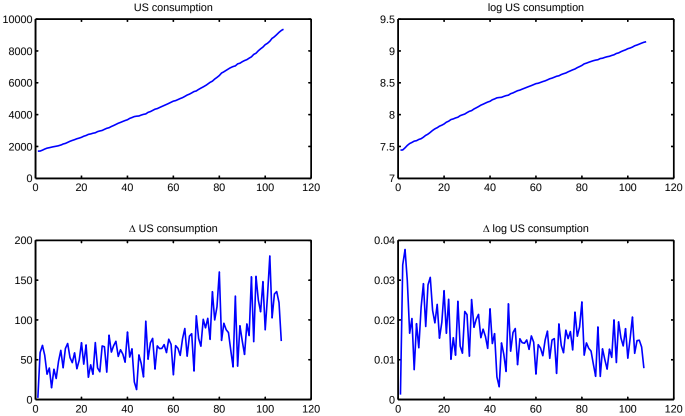
- $\hat{z}^o_{\tau+h|\tau}$ superior to $\hat{z}_{\tau+h|\tau}$ in general. There are exceptions.
- Specification uncertainty can make $\hat{z}_{\tau+h|\tau}$ superior to $\hat{z}^o_{\tau+h|\tau}$

Overview

Nonlinear transformations

The log transformation

- log transformations are common in time series analysis.
- Reasons:
  - Linearize economic relations.
  - Stabilize variance of a time series.
  - Make model residuals look normal.
- Question of interest:
  Is the log transformation useful for forecasting?


US consumption
10000
8000
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log US consumption
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Predictors

Linear predictor

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$y^{nai}_{t+h|t} = \exp x_{t+h|t}$,
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Optimal predictor

$y^{opt}_{t+h|t} = \exp(x_{t+h|t} + \frac{1}{2}\sigma^2_x(h))$,
where $x_{t+h|t}$ is based on ARMA model for $x_t = \log y_t$ and $\sigma^2_x(h)$ is the corresponding forecast error variance.

Table 1: Forecast MSEs (Naive/Linear) for Consumption Series

Country	h	Sample 1980Q1- Forecast period 2000Q1	Sample 1980Q1- Forecast period 2003Q1	Sample 1985Q1- Forecast period 2000Q1	Sample 1985Q1- Forecast period 2003Q1	Sample 1990Q1- Forecast period 2000Q1	Sample 1990Q1- Forecast period 2003Q1
US	1	1.0235	0.7678	1.0685	0.6963	1.1045	0.8015
	2	0.9416	0.4251	0.9251	0.4385	0.9587	0.5327*
	3	0.8956	0.2590	0.8348	0.3175	0.8810	0.4467
	4	0.8437	0.2879	0.7458	0.3156	0.7302	0.4228

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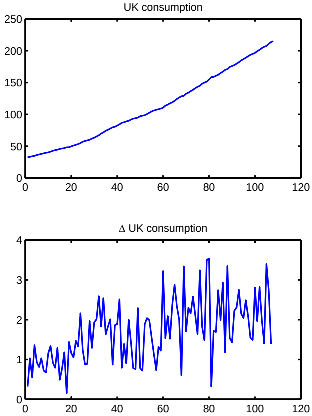
UK	1	1.8154	2.0701	1.8468	2.0323	1.5834	2.5313
	2	3.3214*	3.3397	3.5009	2.5500	2.7726*	3.1062
	3	4.0529*	5.7324*	4.1098*	3.3790*	3.0556*	3.6966
	4	5.2223*	7.8226*	5.1315*	4.6133*	3.7520*	4.6226*

Note: AR order selection based on SC with maximum lag order of 8.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Problems related to aggregation

- Theoretical results relate to linear aggregation
- Forecast aggregate or aggregate forecasts?
- Which forecast is optimal?

Non-Gaussian processes

- Linear forecasts may not be best (minimum MSE) anymore
- Use, e.g., bootstrap methods to compute interval forecasts

  Findley(1986), Masarotto (1990), Grigoletto (1998), Kabaila (1993), Kim (1999), Clements and Taylor (2001), Pascual, Romo and Ruiz (2004)

Aggregation with time-varying weights

EU unemployment rate

$$u_t^{EU} = \sum_{i=1}^{N} w_i u_t^{(i)}$$

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EMU growth rate (Winder 1997)

$$\Delta \log y_t^{EU} = \sum_{i=1}^{N} \frac{y_{t-1}^{(i)}/e_{TB}^{(i)}}{y_{t-1}^{EU}} \Delta \log y_t^{(i)}$$

where $e_t^{(i)}$ is the exchange rate of country i

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Structural change

Problem in constructing pre-EMU data:
Adjustment processes in some countries during run-up to EMU possibly induced structural changes in generation process of some variables

Possible solution:
Use German data for pre-EMU period
(Brüggemann and Lütkepohl, 2005, 2006)

Adjusting German data to EMU level

- No adjustment (GER0): West-German (70Q1-90Q4), German (91Q1-98Q4) and EMU data (99Q1-) have been joined

- Statistical adjustment (GER1): GER0 data has been increased to EMU level in 99Q1. German series has been adjusted multiplicatively using factor:

  $x = \frac{y_{99Q1}^{\text{EUR}}}{y_{99Q1}^{\text{GER0}}}$

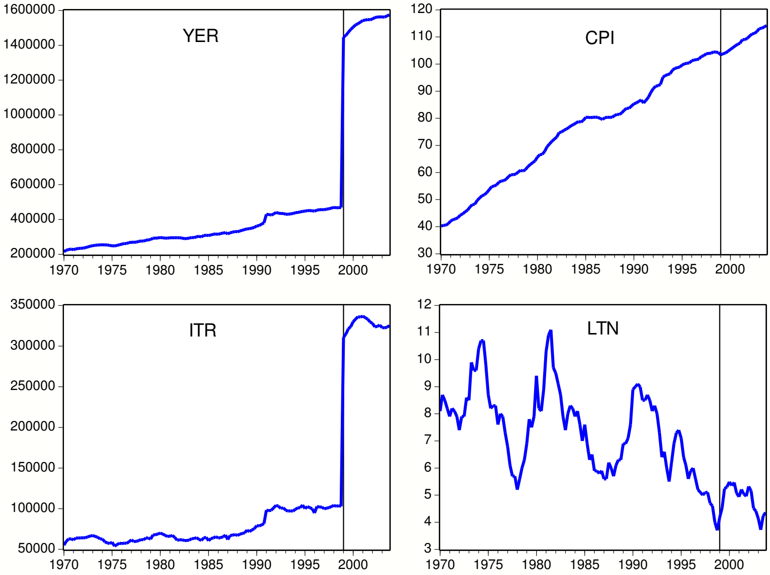

Figure 3: Time series based on German data: GER0

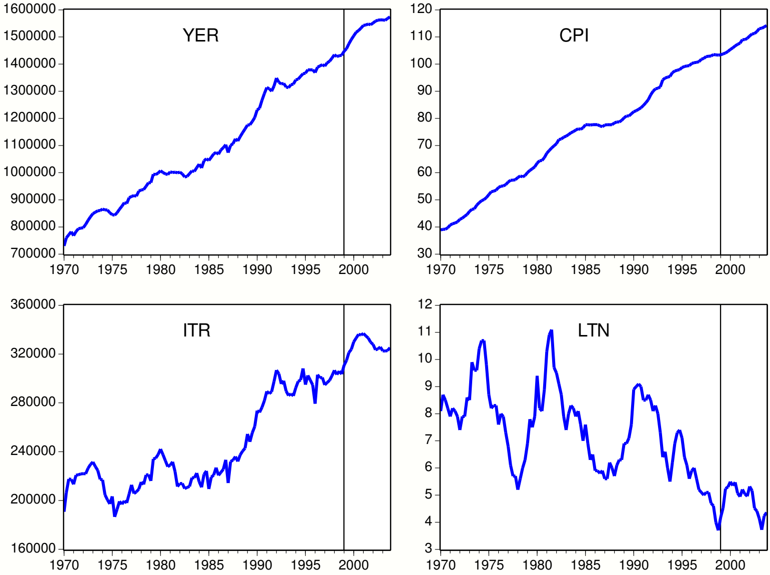

Figure 4: Time series based on German data: GER1

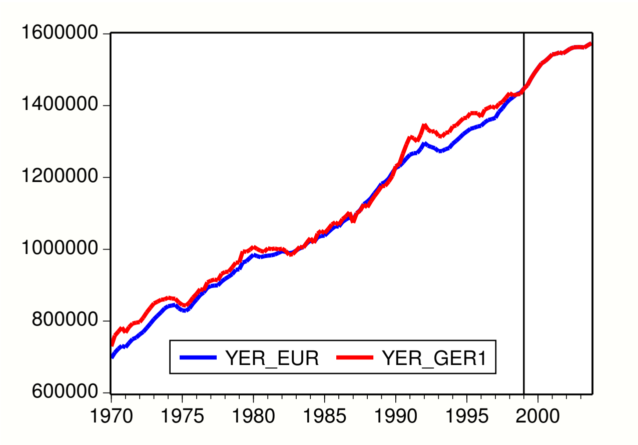

Figure 5: Real GDP based on German and European data

Forecast comparison

	Models used
AR1	AR(4) with constant, levels
AR2	AR(4) with linear trend, levels
AR3	AR(4) with constant, first differences
AR4	AR(4) with linear trend, first differences
AR5	AR(4) with constant, pretest for unit root
AR6	AR(4) with linear trend, pretest for unit root
AR7	AIC-AR with constant, levels
AR8	AIC-AR with linear trend, levels
AR9	AIC-AR with constant, first differences
AR10	AIC-AR with linear trend, first differences
AR11	AIC-AR with constant, pretest for unit root
AR12	AIC-AR with linear trend, pretest for unit root
AR13	BIC-AR with constant, levels
AR14	BIC-AR with linear trend, levels
AR15	BIC-AR with constant, first differences
AR16	BIC-AR with linear trend, first differences
AR17	BIC-AR with constant, pretest for unit root
AR18	BIC-AR with linear trend, pretest for unit root

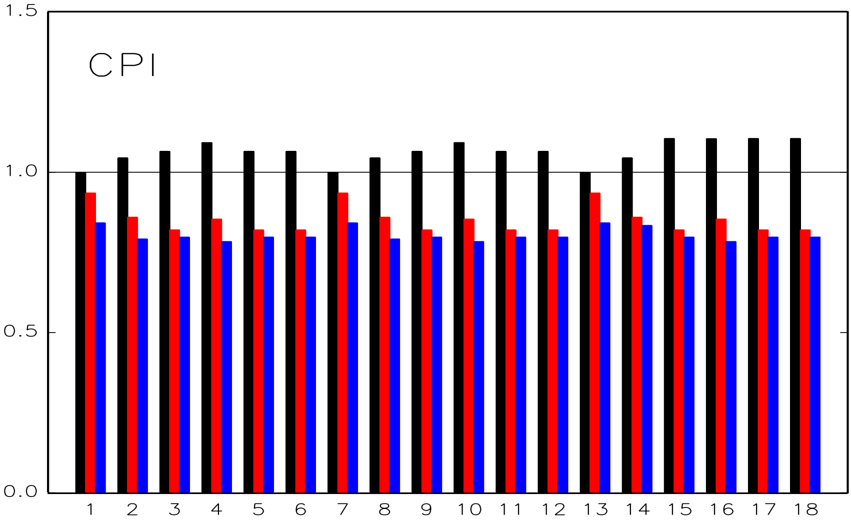

Figure 6: MSEs relative to the AR benchmark model for aggregated Euro-area data ($h = 1$). Black: EUR, Red: GER1, Blue: GER2

Overview

Conclusions

Predictors for contemporaneous aggregates:

- Direct forecast of aggregated variable
- Forecast disaggregated process and aggregate forecasts (is theoretically optimal)
- Forecast component series individually and aggregate forecasts

Many results are available for linear aggregates when the DGPs are known.

Conclusions

Practical problems for linear aggregates:

- Parameter estimation
- Model specification

More general practical problems:

- Nonlinear transformations, e.g., log
- Non-Gaussian processes
- Time-varying aggregation weights
- Structural change