Real-Time Filtering: Using the Multivariate DFA to Monitor the US Business Cycle

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November 2008
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Abstract

Real-time filtering is defined as the estimation of signals near the end of a data sample. Especially for current policy issues and now- and forecasting activities the need for extracting accurate signals at the end of the sample period is evident. This paper builds on the so-called Direct Filtering Approach (DFA) which explicitly accounts for real-time signal extraction problems (Wildi, 1998). In an univariate set-up, it has already been shown that this approach, which uses optimization criteria defined in the frequency domain, can lead to substantial real-time performance improvements as compared to established linear and newer non-linear approaches (see, e.g. http://www.neural-forecasting-competition.com/results.htm).

We extend the DFA to multivariate filtering and apply it to monitor business cycle developments in the US in real-time. Besides real GDP, this multivariate approach allows us to include information on industrial production, capacity utilization, non-farm output, business sector employment, hours worked and help wanted adds. Especially when allowing for the actual publication delays in national accounts data, this new method outperforms other recently developed univariate and multivariate filtering approaches.

Keywords: Real-time signal extraction, multivariate filtering, US Business Cycle
JEL classification: C32, C61, E32, E37

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1 Introduction

To understand business cycle developments and produce short- to medium-term forecasts many methods are available nowadays. Signal extraction via the use of different filter approaches is often part of any of these methods. For example, business cycle filters such as those proposed by Hodrick and Prescott (1980) or Baxter and King (1995) are widely used in applied business cycle analysis. Especially for forecasting purposes the performance of these and other filters are near the sample endpoints often far from optimal. Orphanides and van Norden (2002) illustrate some of the difficulties encountered in the context of real-time output-gap measures and Wildi (2008b) analyzes real-time trend extraction problems in the context of leading indicators.

A business cycle describes the fluctuations in macroeconomic variables typical for market economies, fluctuations which, despite all idiosyncrasies and without any marked periodicity, follow a given pattern and are correlated across those variables. Hence, instead of focusing on one time series, other approaches have emerged which distill information from several variables to extract the underlying signal of the factor in which the researcher is interested. For instance, Dynamic Factor Modelling (DFM), as investigated by authors such as Forni and Reichlin (1998), Forni et al. (2000, 2005) and Stock and Watson (2002), has become a rather popular transversal filtering technique. The use of additional variables facing the same underlying signal might help reduce the noise present in the series targeted. On top of that, some of these variables might be valuable as they are leading over others during the business cycle. Furthermore, if the reference series faces a publication lag and is prone to subsequent data revisions, coincident indicators which are less prone to revisions and have shorter publication lags can contain useful information as well.

Despite the various attempts which have been made to improve filter performance in real-time settings, what seems largely lacking are methods and studies that integrate transveral and longitudinal filtering approaches. In the framework of univariate designs, approaches proposed by Christiano and Fitzgerald (2003), Schleicher (2003) and van Norden (2004) minimize filter distortions by customizing the filter to match the dynamics of the series to be filtered. This is also called the Direct Filter Approach (DFA) and was independently introduced in Wildi (1998). As further developed in Wildi (2004, 2008b) these filters allow for an efficient trade-off between speed (in terms of filter lag) and reliability (in terms of the expected deviation from the true signal).

In the domain of multivariate filter designs recent advances are proposed by Valle e Azevedo et al. (2006) and Valle e Azevedo (2008). They propose a multivariate generalization of the Christiano-Fitzgerald (CF) filter and thereby are in principle confronted with the same kind of end point and efficiency problems as the CF-filter itself. Recently Wildi (2008a) has proposed a multivariate generalization of the DFA called MDFA.

In this paper we use this MDFA to extract the underlying signal of real GDP in the US in real time. As largest economy in the world it has substantial influence on global economic developments and is therefore in the focus of most business cycle researchers around the world. Besides real GDP our data set contains industrial production, capacity utilization, non-farm output, hours worked in the business sector, average weekly hours worked and the Help Wanted Ads Index. Besides their sensitivity to the underlying business cycle, these series are published well in advance of real GDP data. As we will show, the MDFA will exploit these benefits.

The next section will explain the (M)DFA as proposed by Wildi (2008a). We will briefly go into the other more familiar filtering approaches which we will apply in the empirical part of this paper in Section 3. Section 4 describes the adopted experimental framework. Section 5 applies the method and compares the results of the MDFA to those of other methods like the multivariate CF filter as proposed by Valle e Azevedo (2008) in a set-up in which it is assumed that all time series are available at the same time. Subsequently, a similar excercise is done in which it is somewhat more realistically assumed that GDP is released with a publication lag. We end with some concluding remarks.

2 (Univariate) DFA and (Multivariate) MDFA

Instead of first designing a filter based on some *ad hoc* criterion and then afterwards checking the efficiency of the filter in the dimensions needed, the Direct Filter Approach starts by having efficiency explicitly enter the design of the optimization criterion. It thereby can account for the user preferences and research purpose. In our context, we assume that the user is interested in approximating a particular signal in a 'mean-square' sense[1], which is to be precised now.

In general, the signal can be defined by the output of a (bi-infinite) filter

$$Y_t = \sum_{|k|<\infty} \gamma_k X_{t-k}$$

where X_t is a bi-infinite realization of the original time series. Of course, in practice, only a finite sample $X_1, ..., X_T$ is observed. Moreover, users are often interested in obtaining information about the signal towards the endpoint $t = T$ of the sample. The corresponding 'real-time' estimation

[1]More general criteria are available if turning-points are of concern, see Wildi (2004), (2008b).

problem then consists in finding filter-weights $\hat{\gamma}_k$, $k=0,...,T-1$ such that the real-time estimate

$$\hat{Y}_T=\sum_{k=0}^{T-1}\hat{\gamma}_k X_{T-k}$$

minimizes the mean-square filter error

$$E[(Y_T-\hat{Y}_T)^2]$$

If X_t is stationary, then Wildi (1998) proposes the following optimization criterion

$$\min_{\{\hat{\gamma}_k\}}\ \frac{2\pi}{T}\sum_{k=-(T-1)/2}^{(T-1)/2}|\Gamma(\omega_k)-\hat{\Gamma}(\omega_k)|^2 I_{TX}(\omega_k) \qquad (1)$$

where $\Gamma(\omega)$, $\hat{\Gamma}(\omega)$ are the transfer functions of the symmetric bi-infinite and the asymmetric finite filters, $I_{TX}(\omega_k)$ is the periodogram of X_t and $\omega_k=k2\pi/T$, $k=0,...,(T-1)/2$ is a discrete frequency support[2]. In general, estimating all coefficients $\hat{\gamma}_k$, $k=0,...,T-1$ simultaneously would lead to overfitting. Therefore we parameterize real-time filters in the form of particular ARMA-designs, (see Appendix 7).

The above expression is a finite sample approximation of the following integral

$$\int_{-\pi}^{\pi}|\Gamma(\omega)-\hat{\Gamma}(\omega)|^2 dH(\omega)=E[(Y_t-\hat{Y}_t)^2] \qquad (2)$$

where $H(\omega)$ is the spectral distribution of X_t[3]. In order to point towards *efficiency*, it is necessary to decompose the approximation error linking (1) to (2):

$$\begin{aligned}E[(Y_t-\hat{Y}_t)^2] &= \frac{1}{T}\sum_{t'=1}^{T}(Y_{t'}-\hat{Y}_{t'})^2+e_T &(3)\\ &= \frac{2\pi}{T}\sum_{k=-T/2}^{T/2}|\Gamma(\omega_k)-\hat{\Gamma}(\omega_k)|^2 I_{TX}(\omega_k)+e_T+e'_T &(4)\end{aligned}$$

Wildi (2004) , Chapter 5, and Wildi (2008b) , Chapter 3, show that e_T, in the first stage, is asymptotically smallest possible and that e'_T, in the second stage, is of smaller magnitude than e_T. Moreover, it is shown that these characteristics of the estimate apply uniformly under suitable regularity assumptions. Formally, 'efficiency' then means that the solution of (1)

[2] For notational simplicity we assume throughout the paper that T is an odd integer.

[3] In general (eg in most practically relevant applications) the filter error $Y_t-\hat{Y}_t$ is stationary so that $E[(Y_t-\hat{Y}_t)^2]$ does not depend on t.

minimizes a superconsistent estimate of an efficient estimate of the mean-square filter error (2). It seems plausible to interpret part of the observed efficiency gains in practical applications as a consequence of the immanent efficiency of the approach[4]. More general criteria involving non-stationary (integrated) processes or emphasizing turning-points are provided in Wildi (2008b) , chaps. 5 and 6.

A formal generalization of these results to multivariate filters is provided in Wildi (2008a). The main idea is to carry over the immanent efficiency of the univariate approach to a multivariate setting. In order to illustrate some of the underlying concepts we assume the existence of additional 'explanatory' time series W_{it}, $i = 1, ..., m$ which are susceptible to convey useful information about the signal Y_t beyond that already contained in X_t. Let the output of the multivariate real-time filter be defined by

$$\sum_{k=0}^{T-1} \hat{\gamma}_{0k} X_{t-k} + \sum_{k=0}^{T-1} \hat{\gamma}_{1k} W_{1t-k} + ... + \sum_{k=0}^{T-1} \hat{\gamma}_{mk} W_{mt-k}$$

As can be seen, the latter expression generalizes univariate filters - by distilling information from additional 'explanatory' variables $W_{it}, i = 1, ..., m$ - and it generalizes dynamic factor models by extending the purely transversal filtering - obtained by the aggregation operator - to a *simultaneous longitudinal* as well as *transversal* filtering: both dimensions of the index of $\hat{\gamma}_{ik}$, $i = 0, .., m,\ k = 0, ..., T-1$ are addressed.

It is important to briefly emphasize the underlying fundamental issues by relying on a well-known practical problem which is sometimes termed as "direct vs. indirect filtering", see Astolfi et al.(2001) . The authors rise the question whether "the European indicator is first computed by aggregation of the raw data and then seasonally adjusted" or "the raw data (for example the data by country) are first seasonally adjusted and the European seasonally adjusted series is derived by aggregation of these series". Obviously, this problem addresses difficulties when relying on separate - sequential - transversal and longitudinal filters. The great advantage - power - of the proposed general multivariate (real-time) filter MDFA is to allow for a *simultaneous* and *efficient* filtering in both dimensions of the estimation problem.

The main theoretical result is reported in section Appendix 7 and some of the relevant issues associated to the derived optimization criteria are briefly discussed there as well.

[4]The DFA outperformed X-12-ARIMA, TRAMO/SEATS, CF- and HP-filters as well as a whole bunch of forecasting approaches - including winner and runner-up of the prestigious M3 competition -, see Wildi (2008b) and the outcomes of the NN3 and NN5 forecasting competitions http://www.neural-forecasting-competition.com/results.htm.

3 Some remarks on real-time filtering methods

In order to rely on a common general framework we note that

$$Y_T = \sum_{|k|<\infty} \gamma_k X_{T-k}$$

may be approximated by

$$\hat{Y}_T = \sum_{-\infty}^{-1} \gamma_k \hat{X}_{T-k} + \sum_{0}^{T-1} \gamma_k X_{T-k} + \sum_{T}^{\infty} \gamma_k \hat{X}_{T-k}$$

Thus the estimation problem may be tackled in the time-domain by supplying optimal estimates - forecasts and/or backcasts - $\hat{X}_{T-k}$ of the data. X-12-ARIMA and TRAMO/SEATS for example rely on (Reg-)ARIMA-forecasting models whereas STAMP relies on state-space models. A comparison of these approaches with the (univariate) DFA can be found in Wildi (2008b) , Chapters 3 and 4.
For benchmarking the (multivariate) MDFA we here rely on a setting proposed in Valle e Azevedo (2008). Besides a multivariate extension of the CF-filter, the author of this study also considers (univariate) HP- and CF-filters[5]. CF- and HP-filters rely on particular assumptions about the data-generating process which enable to derive optimal fore- and/or backcasts, given that the underlying model assumptions are satisfied. A common implementation of the CF-filter relies on the assumption that the input series is a random-walk process[6] whereas for the HP-filter an integrated random-walk process (+noise) is assumed. In contrast, the (M)DFA does not rely on a specific model of the data-generating process. Instead, the information is summarized in a sufficient statistic - the periodogram - of the data. Therefore, it is more flexible than common implementations of CF- and HP-filters. Moreover, the optimization criterion (1) emphasizes 'directly' the filter error and therefore it emphasizes - implicitly - a particular linear combination of all fore- and backcasts simultaneously. This is to be contrasted with X-12-ARIMA or TRAMO/SEATS which focus on short-term one-step ahead forecasting performances exclusively. Methodological differences between the multivariate CF-filter and the MDFA are discussed in Wildi (2008a), section 5.

4 Experimental Design

We are interested in the business cycle development of the US economy. Real GDP is in general considered to be the time series most closely reflecting general economic developments of a nation. On top of that, we have

[5]Other designs such as Baxter and King or Butterworth filters were rejected because, according to the author, they were not explicitly designed for real-time applications.

[6]Christiano and Fitzgerald (2003) recommend that setting for unit-root processes.

selected the industrial production index, capacity utilization in the industry, non-farm output, hours worked in the business sector, average weekly hours and the help wanted ads index. Stock and Watson (1999) and Valle e Azevedo (2008) find that these indicators not only are highly affected by the business cycle but in general appear to have a lead w.r.t. developments in real GDP. Furthermore, as compared to real GDP, these series have the advantage that their publication lag is in general much shorter. All data are in (or have been transformed to) quarterly frequency, start in the first quarter of 1967 and end in the second quarter of 2005[7]. Fig. 2 shows the log-transformed and de-trended data (for sake of illustration all series are standardized in the plot). The overall picture does not suggest that there is a systematic lead for all of these series. Some nevertheless exhibit anticipative behavior around turning points. Furthermore, smaller publication lags might create an information lead of many of these series in real-time.

It is common practice to define the business cycle frequency to be between 6 and 32 quarters. Inline with this, we have filtered our target variable, i.e. de-trended log-GDP over the full sample, with a classic 'symmetric' univariate CF-filter (6-32 quarters)[8]. The resulting filter output is our signal Y_t that we have to approximate by the real-time designs. Since the 'symmetric' CF-filter becomes more and more asymmetric towards the end point we disregard observations at the beginning and the end of the sample (see next section for details).

The most important advantages of relying on a pre-existing experimental framework, as proposed by Valle e Azevedo (2008), are that we 'only' have to complement existing results by new ones based on (M)DFA - thus comparisons are straightforward and unbiased - and that results can be cross-validated. A possible weakness is to be seen in the imposed structure of the experiment. So for example, the proposed design is a 'pseudo' real-time exercise in the sense that data revisions are ignored and that reported results are based on 'whole sample' estimates[9]. In order to widen up the scope of our analysis we here report results for 'whole-sample' as well as out-of-sample experiments and we allow for publication lags of the GDP series.

Finally, in order to stick closely to the pre-existing empirical framework, we here ignore issues related to potential cointegration amongst the original time series. Hence, the DFA results are based on criterion (1) for the univariate case or (7) in the multivariate case.

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[7]More detailed descriptions of the data can be found in Appendix 7.

[8]Strictly speaking, the filter is symmetric in the middle of the sample only and becomes more and more asymmetric towards the boundaries.

[9]Valle e Azevedo (2008) p.27 argues: "The needed second order moments will be the ones obtained using the whole sample. It is expected that the variation stemming from second moments uncertainty will be reduced as the sample size gets larger, i.e., from today onwards. We have however verified that the results are only slightly worse for both filters if we estimate the moments in real-time".

5 Empirical results when neglecting publication lag issues

Table 1 summarizes performances of the various real-time filters on the predefined time span 1972-2002. Correlations are taken between real-time estimates and the signal. Numbers highlighted by an asterisk were originally obtained by Valle e Azevedo (2008, Table 5, p.28). The accordance of our

	MSE's	Corr	MSE-Ratios (MDFA-reference)
HP filter	0.410	0.5*	3.13
CF	0.21	0.77* (0.77)	1.66
CF Multivariate	0.179	0.82*	1.37
DFA	0.209	0.78	1.63
MDFA	0.131	0.872	1

Table 1: Performance Measures 1972-2002

empirical framework with that in Valle e Azevedo (2008) is confirmed by the identical performances obtained for the real-time CF-filter (our result is in parentheses). Since (M)DFA-filters are based on mean-square error criteria we complement the existing results by mean-square performances. More precisely, ratio's of MSE's (for CF multivariate, CF and HP) can be obtained by applying the following transformation to the existing correlation measures

$$\frac{1-\text{cor}_i^2}{1-\text{cor}_j^2}=\frac{\text{MSE}_i}{\text{MSE}_j} \tag{5}$$

where the indices i, j discriminate two competing methods[10].

As can be seen, the MDFA substantially outperforms all competing methods. It is worth to emphasize that this result is obtained in a framework which tends to penalize the (M)DFA. Indeed, the boundaries $2\pi/(6*4)$ and $2\pi/(32*4)$, corresponding in the frequency-domain to our time-domain specification (6-32 quarters), fall between consecutive ω_k's, in (1), and therefore the 'control' of the optimization criterion on the shape of the real-time filter is to some extent blurred[11]. In order to get a feeling of the magnitude of this blurring-effect we compared the traditional CF-filter with (M)DFA based on the band-pass definition 5.8-38 quarters whose boundary frequencies $2\pi/(5.8*4)$ and $2\pi/(38*4)$ in the frequency-domain correspond to ω_4

[10]This transformation is valid under assumptions about the estimates which seem to be satisfied here, see Valle e Azevedo (2008), p.17, for details. The close correspondence between the ratios of the observed MSE's (for 'DFA' and 'CF Unit Root') and the results obtained from 5 confirm the validity of the latter expression.

[11]A formal solution for that problem might be based on section 6.5 in Wildi (2004) .

and ω_{26} in (1). Table 2 summarizes the relevant results on two different time spans (both proposed by Valle e Azevedo (2008)). The magnitude

	MSE (72-02)	Corr (72-02)	MSE(88-02)	Corr (88-02)
CF	0.298	0.72	0.124	0.61
DFA	0.254	0.76	0.057	0.77
MDFA	0.186	0.84	0.034	0.89

Table 2: Performance Measures 1972-2002 and 1988-2002

of the gain of the (univariate) DFA over the CF-filter now corresponds to previous results obtained for X-12-ARIMA and TRAMO, see Chapter 4 in Wildi, 2008. The sensitivity of the (absolute) performance measures to the time span definition can be appreciated in the above table. A comparison of Tables 1 and 2 highlights the sensitivity of the (absolute) performance measures to 'small' changes of the signal-definition. Note, however, that the ranking (the relative performances) is robust and does not seem to be affected by the above sensitivity analysis.

In order to assess real-time performances in a practically more relevant framework we also estimate filter coefficients of the MDFA on a shorter time span of 21 years, from Q1-1967 to Q4-1987 and compute true out-of-sample filter errors. A comparison of in- and out-of-sample performances is to be found in Table 3. As can be seen, out-of-sample the MDFA barely loses

	MSE(88-02)
MDFA (in sample)	0.034
MDFA (out of sample)	0.041

Table 3: Performance Measures

performance. This confirms previous findings on the DFA (see Chapters 7 and 8 in Wildi (2008)).

Finally, amplitude and time-shift[12] functions of DFA- and MDFA-filters are, together with the band-pass definition, plotted in Figs. 4 to 7. As can be seen, the above estimation problem is challenging because the filters have to sharply discriminate two neighboring low-frequency spectral peaks, the first one being in the stop-band and the second one being in the pass-band of the symmetric band-pass filter. The multivariate filter tackles the problem by completely removing the peak in the stop-band and then picking-up selectively spectral mass from the explanatory variables to 'reconstruct' the signal, see Fig. 4. This is to be contrasted with the DFA-filter in Fig. 5

[12]Phase divided by frequency, see Chapter 3 in Wildi [Wildi, 2008b].

which cannot entirely remove the first peak in the stop-band and tends to damp the second peak in the pass-band. These differences explain the improvement provided by the multivariate design.

6 Empirical results allowing for a publication lag in GDP

In the previous section we have simply used all time series up to the point in time which we want to analyze. In that sense the set-up missed an important attribute which is relevant in real life: the different time series all face different publication lags.[13] Therefore, we now analyze how the results are affected in case we lag (final) GDP by two or four quarters while keeping the other indicators 'in time'. Although this experiment is still not fully comparable with actual practice, it nevertheless points to some relevant issues. In the following we focus on efficient univariate and multivariate designs only - DFA and MDFA - in order to illustrate the potential of multivariate filters in the context of 'publication lags'.[14] Our comparisons will be based on the bandpass definition 5.8-38 quarters which allows for tighter control of the optimization criterion on the real-time filters.

Figures 10 and 11 plot DFA and MDFA outputs for delays zero, two and four quarters. Whereas the univariate filter performs quite well with GDP lagged by two quarters it seems to be affected by becoming more noisy and showing a delayed signal when using four lags. In contrast, the multivariate filter seems to be remarkably immune to the above alterations as can, for example, be seen in the important turning-points. The performances are summarized in Tables 4 (lag 2) and 5 (lag 4) which have to be compared to Table 2 (lag 0).

	MSE (72-02)	Corr (72-02)	Corr (88-02)
DFA	0.277	0.74	0.67
MDFA	0.190	0.84	0.87

Table 4: Performance Measures: GDP lagged by 2 quarters

Figures 14 and 15 compare filter characteristics of the (univariate) DFA for GDP delayed by zero, two and four quarters. As can be seen, part of the degradation in the univariate performances are due to poorer stop-band properties. However, the main loss incurred by delaying GDP by four quarters is due to poorer time-shift properties which cannot compensate the

[13] Another aspect is that most time series are revised over time. Although potentially important, we do not address that issue in this paper for lack of space.

[14] Note that the following results cannot be compared anymore with those obtained in Valle e Azevedo (2008).

	MSE (72-02)	Corr (72-02)	Corr (88-02)
DFA	0.401	0.59	0.51
MDFA	0.198	0.83	0.85

Table 5: Performance Measures: GDP lagged by 4 quarters

lag anymore. Figures 12 and 13 reveal that the multivariate filter slightly shifts the weighting of input signals by dampening GDP, as the lag augments, and by increasing the importance of the other variables (in particular 'HelpWanted' and 'NonFarm'): with increasing lag the filter synthesizes the signal by relying on spectral mass of the explanatory variables which are not affected by publication lags. This effect illustrates impressively the simultaneous longitudinal and transversal filtering ability of the MDFA.

To summarize, the multivariate filter does not seem to be affected by delays of GDP up to four quarters. For the univariate filter, the limit of the anticipation provided by the difference filter is attained for delays of two quarters. From then on, performances of the univariate design deteriorate markedly. The MDFA with GDP delayed by four quarters still beats by a fair margin all other approaches based on GDP available up to the last available time-point.

7 Concluding remarks

This paper has applied different filtering methods on US data to find out which of these methods are best capable to extract the business cycle in real time. We have highlighted that nowadays there are methods available which explicitly optimize generalized forecasting performances - as addressed by real-time signal extraction problems - and thereby outperform more commonly used approaches. The Direct Filtering Approach by construction has efficiency in mind.

Multivariate real-time filters generalize univariate designs as well as dynamic factor models by allowing for simultaneous longitudinal and transversal filtering. The immanent 'directness' of the MDFA, addressing the mean-square filter error explicitly, enables to develop the full potential of the method in particular in the presence of publication lags.

Due to the particular experimental design, a 'horse-race' between competing approaches in a pre-existing empirical framework, some aspects of the real-time signal extraction problem could not be addressed comprehensively. In particular, data revisions were tackled only indirectly, by allowing for publications lags of the GDP series, revision effects due to seasonal adjustment were neglected and long-term equilibria among the time series - cointegration relations - were ignored. These issues are the topic of future work in the domain.[15]

Appendix

Data

- Industrial Production Index (IPI), monthly series - quarterly series is constructed as average of the three months of each quarter. Available from the Board of Governors of the Federal Reserve system. Series ID: INDPRO.
- Capacity Utilization (Total Industry), monthly series - quarterly series is constructed as average of the three months of each quarter. Available from the Board of Governors of the 43 Federal Reserve system. Series ID: TCU
- Non-Farm Output, quarterly series, seasonally adjusted- Available from the U.S. Department of Labor: Bureau of Labor Statistics. Series ID:OUTNFB
- Business Sector: Hours of All Persons, quarterly series, seasonally adjusted. Available from the U.S. Department of Labor: Bureau of Labor Statistics. Series ID:HOABS
- Average weekly hours, monthly series, seasonally adjusted. quarterly series is constructed as average of the three months of each quarter. Available from the U.S. Department of Labor: Bureau of Labor Statistics. Series ID:AWHNONAG
- Help Wanted Ads Index, monthly index, seasonally adjusted - quarterly series is constructed as average of the three months of each quarter. Available from the Conference Board. Series ID:HELPWANT

Filter Design: ZPC-Filters

In our applications all (M)DFA-filters are based on the following general structure:

$$(1-B)(c_1+c_2B+c_3B^2)\frac{\sum_{j=0}^{8} b_j B^j}{1-\sum_{k=1}^{8} a_k B^k}$$

The difference filter induces a vanishing amplitude function of the real-time band-pass filter in frequency zero, the MA(3)-filter accounts for possible

[15]The available data was already seasonally adjusted. Note that the MDFA would perform seasonal adjustment simultaneously with the band-pass filtering and more efficiently than traditional univariate methods (X-12-ARIMA, TRAMO/SEATS).

lags/leads among the variables (its parameters are positive and they add to one) and the ARMA-filter is a ZPC-design that accounts for the general shape (amplitude/phase) of the filter, see below for details.

The optimized parameters for the MDFA based on the 'whole sample' information, when GDP is not lagged, are:

	GDP	Bus.hours	Nonfarm	CapUti	HelpWantedAd	IPI
c_1	0.29593	0.62014	0.03815	0.04383	0.35027	0.20913
c_2	0.50638	0.20438	0.10781	0.02432	0.61577	0.19794
c_3	0.19768	0.17547	0.85402	0.93183	0.03395	0.59291
a_1	-3.600585	-3.293670	-2.642242	-3.6239785	-3.682467	-1.222659
a_2	5.274890	3.602874	2.514952	4.7929967	5.647784	3.6611e-01
a_3	-4.131813	-0.882473	-0.813251	-2.5165765	-4.849347	1.0343e-02
a_4	2.025209	-0.828907	-0.087014	0.1583503	2.849718	9.4900e-04
a_5	-0.734243	0.375651	-0.044521	0.1939470	-1.314755	1.7558e-05
a_6	0.190875	0.040299	0.102525	-0.0037432	0.400838	1.2851e-06
a_7	-0.023710	-0.015465	-0.011815	0.0004320	-0.050777	1.7778e-08
a_8	0.002128	0.005103	0.001361	0.0005137	0.004761	9.2517e-10
b_0	0.557441	6.8260e-02	1.4062e-01	9.8188e-02	7.8344e-02	1.4597e-09
b_1	-1.868924	-1.3971e-01	-2.4909e-01	-2.2097e-01	-2.0083e-01	-9.2739e-10
b_2	2.627830	6.7513e-02	8.5797e-02	1.5118e-01	2.0467e-01	1.0328e-10
b_3	-2.171728	2.0604e-02	2.2175e-02	-2.1900e-02	-1.3141e-01	-4.9525e-10
b_4	1.280729	-1.7696e-02	1.0388e-02	-6.3310e-03	7.0396e-02	2.4200e-10
b_5	-0.552715	1.8430e-03	-6.7856e-03	1.6531e-03	-2.3911e-02	-5.1184e-11
b_6	0.151383	7.9723e-04	1.6004e-03	-3.6131e-05	4.7304e-03	8.2117e-11
b_7	-0.022743	-2.7220e-04	-1.6616e-04	-1.6767e-05	-5.4256e-04	-2.0338e-11
b_8	0.002154	3.9564e-05	1.0704e-05	3.8705e-06	3.4627e-05	7.2168e-12

Note that the IPI-series did not convey useful information beyond that already contained in the other time series so that the MA-coefficients of the ARMA-filter all nearly vanish (accordingly, the corresponding amplitude function vanishes in Fig. 4). It is also important to note that the above 'time-domain' parameters - twenty per time series - do not correspond to effective degrees of freedom but that they are derived from a transformed set of 'frequency-domain' parameters that address the estimation problem more 'directly'. We now briefly introduce the relevant concepts (for technical details the interested reader is referred to Wildi (2008b), section 3.2).

Consider the following recursive ARMA-filter equation

$$\hat{X}_t = \sum_{k=1}^{Q} a_k \hat{X}_{t-k} + \sum_{k=-r}^{q} b_k X_{t-k} \tag{6}$$

For a real-time filter we impose $r = 0$. Stability means that the roots of the characteristic AR-polynomial $1-\sum_{k=1}^{Q} a_k z^k$ lie outside the unit circle. Invertibility or, equivalently, the minimum-phase property is achieved by requiring all zeroes of the characteristic MA-polynomial to lie outside the unit circle. The transfer function of the ARMA-filter (6) is given by

$$\begin{aligned}\hat{\Gamma}(\omega) &= \frac{\sum_{k=-r}^{q} b_k \exp(-ik\omega)}{1-\sum_{k=1}^{Q} a_k \exp(-ik\omega)}\\ &= C\exp(ir\omega)\frac{\prod_{j=1}^{n}(Z_{2j-1}-\exp(-i\omega))(Z_{2j}-\exp(-i\omega))}{\prod_{k=1}^{n'}(P_{2k-1}-\exp(-i\omega))(P_{2k}-\exp(-i\omega))}\\ &\cdot\frac{\prod_{j=2n+1}^{q+r}(Z_j-\exp(-i\omega))}{\prod_{k=2n'+1}^{Q}(P_k-\exp(-i\omega))}\end{aligned}$$

where $Z_{2j} := \bar{Z}_{2j-1}$, $j = 1, ..., n$ are complex conjugate zeroes, $P_{2k} := \bar{P}_{2k-1}$ are complex conjugate poles, $Z_j, j = 2n+1, ..., q+r$ are real zeroes, $P_k, k = 2n'+1, ..., Q$ are real poles and

$$C := b_{-r}\frac{\prod_{k=1}^{Q} P_k}{\prod_{j=1}^{q+r} Z_j}$$

We now consider a particular restriction linking zeroes and poles of the above filter. The resulting design, first introduced in Wildi (2004), is called Zero-Pole-Combination (ZPC-) filter.

The following ARMA(1,1)-transfer function

$$\hat{\Gamma}(\omega) = C\frac{Z-\exp(-i\omega)}{P-\exp(-i\omega)}$$

is called an elementary ZPC-filter, if $\arg(Z) = \arg(P)$. An ARMA(p,p)-filter $\hat{\Gamma}(\cdot)$ is called ZPC-filter if poles $P_k, k = 1, ..., p$ and zeroes $Z_j, j = 1, ..., p$ can be grouped into pairs $(Z_k, P_{j(k)})$ defining elementary ZPC-filters, where $j(k)$ is a suitable (bijective) renumbering of the poles.

In order to illustrate the purpose of the proposed restriction - ZPC design - amplitude and time shift functions of three real ARMA(2,2)-ZPC filters

$$\frac{(Z-\exp(-i\omega))(\overline{Z}-\exp(-i\omega))}{(P-\exp(-i\omega))(\overline{P}-\exp(-i\omega))}$$

(where thus $\arg(Z) = \arg(P)$) are plotted in Fig. 1. Proposition 3.9 and theorem 3.10 in Wildi (2004) show that the amplitude function of an elementary ZPC-filter has a unique extremum in the common argument $-\lambda$ of zero and pole ($\pi/4$ and $\pi/2$ in the above examples[16]). The extremum is a

[16]The location of the extrema of the composed real filter can slightly deviate because conjugate and non-conjugate pairs interact. However, this potential 'interference' is negligible in most applications.

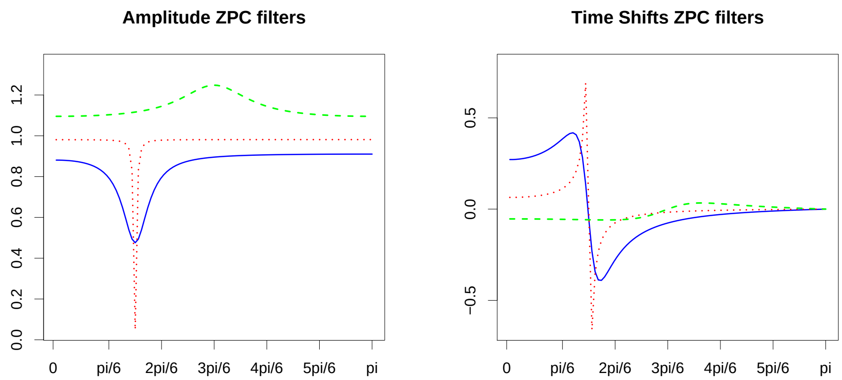

Figure 1: Amplitude and time shift functions of ZPC-filters: $Z = 1.1\exp(-i\pi/4)$, $P = 1.2\exp(-i\pi/4)$ (solid line) and $Z = 1.001\exp(-i\pi/4)$, $P = 1.02\exp(-i\pi/4)$ (dotted line) and $Z = 1.6\exp(-i\pi/2)$, $P = 1.5\exp(-i\pi/2)$ (shaded line)

maximum if the pole is closer to the unit circle (shaded line in Fig.1, left panel), otherwise it is a minimum (solid and dotted lines). The amplitude function in λ is

$$\left|\frac{|Z|\exp(-i\lambda) - \exp(-i\lambda)}{|P|\exp(-i\lambda) - \exp(-i\lambda)}\right| = \frac{|Z|-1}{|P|-1}$$

Therefore, the ratio $(|Z|-1)/(|P|-1)$ controls the height of the extremum: in the above examples 0.5 (solid line), 0.05 (dotted line) and 1.2 (shaded line)[17]. Finally, given the location λ and the extremal value $(|Z|-1)/(|P|-1)$, the width of the trough (peak) can be controlled by the ratio $|Z|/|P|$: for $|Z|/|P| \to 1$ the trough (peak) almost disappears due to canceling zeroes and poles. In the above example, we measure 0.9166 (solid line), 0.9814 (dotted line) and 1.0666 for the corresponding ratios.

Because of their straightforward interpretability, we have adopted the transformed parameter set $p_1 := \lambda$, $p_2 := (|Z|-1)/(|P|-1)$ and $p_3 := |Z|/|P|$. 'Traditional' moduli of zeroes and poles are derived from the latter two parameters. The transformed parameter space (p_1, p_2, p_3) of real ARMA(2,2)-ZPC-filters is designed specifically for matching the location, the height and the width of particular (dominant) spectral peaks. Therefore, the parsimony concept inherent to ARMA-designs becomes strengthened by

[17] These values slightly differ from the plotted extrema because the latter are computed for the *real* ARMA(2,2)-filter: the complex conjugate ZPC-filters disturb marginally the results.

a constraint - the common argument of ZPC-filters - which emphasizes interpretability of parameters. This particular design constrains the available degrees of freedom to match the 'salient features' of the data. As a consequence, (M)DFA real-time filters are immune against overfitting[18], see a comprehensive analysis of the overfitting-problem in Wildi (2008b), Chapters 7 and 8.

MDFA

The following theoretical result corresponds to theorem 7.1 in Wildi (2008a). It summarizes the main results in the context of real-time multivariate filtering under various assumptions about the time series which address the integration order, stationarity or non-stationarity, and in the latter case the cointegration rank of the non-stationary data.

Theorem 7.1 *1. Assume all series are stationary processes admitting absolutely summable Wold decompositions. Then the following statements apply:*

- *The optimization criterion is given by*

$$\frac{2\pi}{T}\sum_{k=-(T-1)/2}^{(T-1)/2}|\Xi_{Tr}^0(\omega_k)|^2 \to min \tag{7}$$

where $\Xi_{Tr}^0(\omega_k)$ is defined in (10). No filter constraints are imposed during optimization. Under mild additional assumptions about $\Gamma(\cdot),\hat{\Gamma}(\cdot)$ and X_t (see proposition 7.3[19] *in Wildi (2008a), stationary case) the resulting optimization corresponds to the minimization of a superconsistent estimate of an asymptotically efficient estimate of the (true unknown) revision error variance.*

- *In the absence of explanatory variables, 7 simplifies to 1.*

2. We assume that all series are integrated with (common) integration order $d = 1$. If the filter coefficients of

$$\Gamma(\cdot),\hat{\Gamma}_X(\cdot),\hat{\Gamma}_{W_1}(\cdot,)...,\hat{\Gamma}_{W_m}(\cdot)$$

[18] As confirmed by our recent results in the framework of the NN3- and NN5-forecasting competitions. Our forecasting models are derived from richly parameterized filters (20 effective degrees of freedom) and substantially outperform traditional 'parsimonious' approaches such as ARIMA-models, exponential smoothing or winner and runner-up of the prestigious M3-competition.

[19] In particular it is assumed that $\sum|\gamma_k||k|^{1/2} < \infty$ and that poles of the real-time ARMA-filter $\hat{\Gamma}(\cdot)$ are larger than $1+\delta$ in absolute value, where $\delta > 0$ is an arbitrary positive number (this restriction can be very easily implemented in practice). Note that $\sum|\hat{\gamma}_k||k|^{1/2} < \infty$ is then automatically satisfied. Since δ does not depend on T, uniformity of efficiency and superconsistency results can be derived. The corresponding regularity assumption is called uniform stability in Wildi (2004).

decay sufficiently rapidly[20]*, then the following statements apply:*

- *If the input series are cointegrated with cointegration vector* $(1, -\alpha_1, ..., -\alpha_m)$ *then the optimization criterion is given by*

$$\frac{2\pi}{T} \sum_{k=-(T-1)/2}^{(T-1)/2} |\Xi'_{Tr}(\omega_k)|^2 \rightarrow min \tag{8}$$

where $\Xi'_{Tr}(\omega_k)$ *is defined in (11). It is assumed that the filter constraints (6) in Wildi (2008a) are satisfied. If the filters* $\Gamma(\cdot), \hat{\Gamma}(\cdot)$ *satisfy a slightly stronger regularity assumption*[21] *and if the first differences of all variables admit absolutely summable Wold-decompositions, then the resulting optimization corresponds to the minimization of a superconsistent estimate of an asymptotically efficient estimate of the (true unknwon) revision error variance.*

- *If the input series are not cointegrated then the optimization criterion is given by*

$$\frac{2\pi}{T} \sum_{k=-(T-1)/2}^{(T-1)/2} |\Xi''_{Tr}(\omega_k)|^2 \rightarrow min \tag{9}$$

where $\Xi''_{Tr}(\omega_k)$ *is defined in (12) and the restrictions* $\Gamma(0) = \hat{\Gamma}_X(0)$ *and* $\hat{\Gamma}_{W_h}(0) = 0$ *are imposed. If we assume the same regularity assumptions as in the preceding statement, then the above expression is well-defined and the same efficiency arguments apply.*

- *If the cointegration space is of dimension larger than one then the optimization criterion is the same as in (8) but the filter constraints (6) in Wildi (2008a) must be replaced by (7) in Wildi (2008a). This implies that one has additional parameters - degrees of freedom - in* $\Xi'_{Tr}(\omega_k)$*. The same efficiency arguments apply as in the preceding two statements.*
- *In the absence of explanatory variables the criterion 9 reduces to the univariate criterion in the case of integrated processes (Formula (3) in Wildi (2008a)).*

The following expressions are referenced by the theorem

$$\Xi^0_{Tr}(\omega_k) \quad := \quad \Delta\Gamma_X(\omega_k)\Xi_{TX}(\omega_k) - \sum_{h=1}^{m} \hat{\Gamma}_{W_h}(\omega_k)\Xi_{TW_h}(\omega_k) \tag{10}$$

where

[20] One requires the stronger assumption $\sum |\gamma_k||k| < \infty$ which is automatically satisfied for the uniformly stable real-time ARMA-filters.

[21] One requires the stronger assumption $\sum |\gamma_k||k|^{3/2} < \infty$.

- $\Delta\Gamma(\cdot) = \Gamma(\cdot) - \hat{\Gamma}(\cdot)$ is the transfer function 'residuum'

- $\Xi_{TX}(\cdot)$ and $\Xi_{TW_h}(\cdot)$ are the discrete Fourier transforms of the data

- $\omega_k = k2\pi/(T-1)$, $k = 0, ..., (T-1)/2$ (we here assume T to be an odd integer).

Furthermore

$$\Xi'_{Tr}(\omega_k) := \Bigg\{ \Delta\Gamma_X(0)\Xi_{TC}(\omega_k) \tag{11}$$

$$- \exp(i\omega_k)\left[\frac{\Delta\Gamma_X(0) - \exp(-i\omega_k)\Delta\Gamma_X(\omega_k)}{1-\exp(-i\omega_k)} - \Delta\Gamma_X(0)\right]^+ \Xi_{T\Delta X}(\omega_k)$$

$$+ \left[\frac{\Delta\Gamma_X(0) - \exp(-i\omega_k)\Delta\Gamma_X(\omega_k)}{1-\exp(-i\omega_k)}\right]^- \Xi_{T\Delta X}(\omega_k)$$

$$+ \sum_{h=1}^{m} \exp(i\omega_k)\left[\frac{\hat{\Gamma}_{W_h}(0) - \exp(-i\omega_k)\hat{\Gamma}_{W_h}(\omega_k)}{1-\exp(-i\omega_k)} - \hat{\Gamma}_{W_h}(0)\right]\Xi_{T\Delta W_h}(\omega_k)\Bigg\}$$

where

- $\Xi_{T\Delta X}(\cdot)$ and $\Xi_{T\Delta W_h}(\cdot)$ are the discrete Fourier transforms of the differenced data and $\Xi_{TC}(\omega_k)$ is the discrete Fourier transform of the stationary process $C_t = X_t - \sum_{h=1}^{m} \alpha_h W_{ht}$ (cointegration residuum).

- $[\sum_{|k|<\infty} \nu_k \exp(-i\omega_k)]^+$ means the positive expansion ($k \geq 0$) of the sum and similarly $[\sum_{|k|<\infty} \nu_k \exp(-i\omega_k)]^-$ means the (strictly) negative expansion ($k < 0$).

Finally

$$\begin{aligned}\Xi''_{Tr}(\omega_k) \quad &:= \quad -\frac{\left(\Gamma(\omega_k) - \hat{\Gamma}_X(\omega_k)\right)}{1-\exp(-i\omega_k)}\Xi_{T\Delta X}(\omega_k) \\ &\quad + \sum_{h=1}^{m}\left\{\frac{\hat{\Gamma}_{W_h}(\omega_k)}{1-\exp(-i\omega_k)}\Xi_{T\Delta W_h}(\omega_k)\right\}\end{aligned} \tag{12}$$

See Wildi (2008a) for a proof of the theorem. We briefly comment the above result from an intuitive point of view:

- Theorem 7.1 shows that the design of the real-time filter optimization criterion depends on the structure of the time series (stationary or integrated, with or without single or multiple cointegration relations).

- Whereas the stationary case can be straightforwardly related to multivariate spectral decomposition theorems (see for example Valle e Azevedo (2008)) the non-stationary case is a little bit more intricate.

– In the case of no cointegration one should note that the expression (12) is well-defined in frequency zero, assuming the filter constraints and regularity assumptions in the theorem are satisfied. Formally, this expression may then be rewritten as

$$\begin{aligned}\Xi''_{Tr}(\omega_k) \quad := \quad & -\Big(\Gamma(\omega_k) - \hat{\Gamma}_X(\omega_k)\Big)\frac{\Xi_{T\Delta X}(\omega_k)}{1-\exp(-i\omega_k)} \\ & + \sum_{h=1}^{m}\left\{\hat{\Gamma}_{W_h}(\omega_k)\frac{\Xi_{T\Delta W_h}(\omega_k)}{1-\exp(-i\omega_k)}\right\}\end{aligned}$$

Therefore, we see that the discrete Fourier transform in the stationary case is replaced by the pseudo-discrete Fourier transform which is intuitively more appealing.

– The expression (11) is more subtle because it involves the cointegration residuum C_t (its discrete Fourier transform). Isolating C_t has some incidences on the expression of the transfer functions of the differenced processes. However, one can show that all transfer functions are well-defined in frequency zero under the assumptions of the theorem. One can see, also, that the filter restrictions in the previous case (no cointegration) can be relaxed since neither $\Delta\Gamma_X(0)$ nor $\hat{\Gamma}_{W_h}(0)$ have to vanish anymore. The price to be paid for that additional degree of freedom is precisely the appearance of the cointegration residuum C_t.

– If the dimension of the cointegrating space increases, then additional degrees of freedom are available for computing optimal real-time filters. If the cointegration rank is identical with the number of variables than all time series are stationary and no filter constraints have to be imposed.

Figures and Graphs

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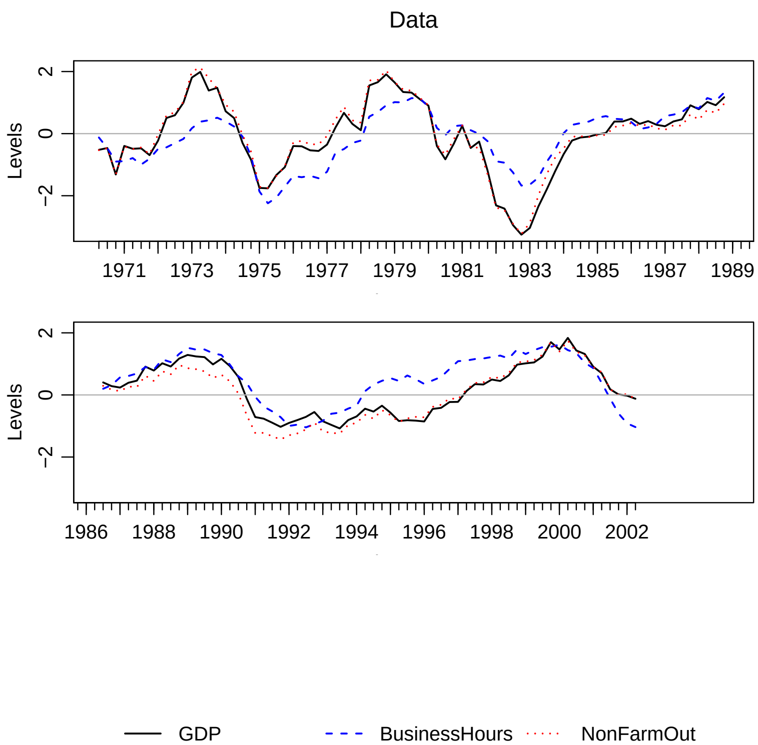

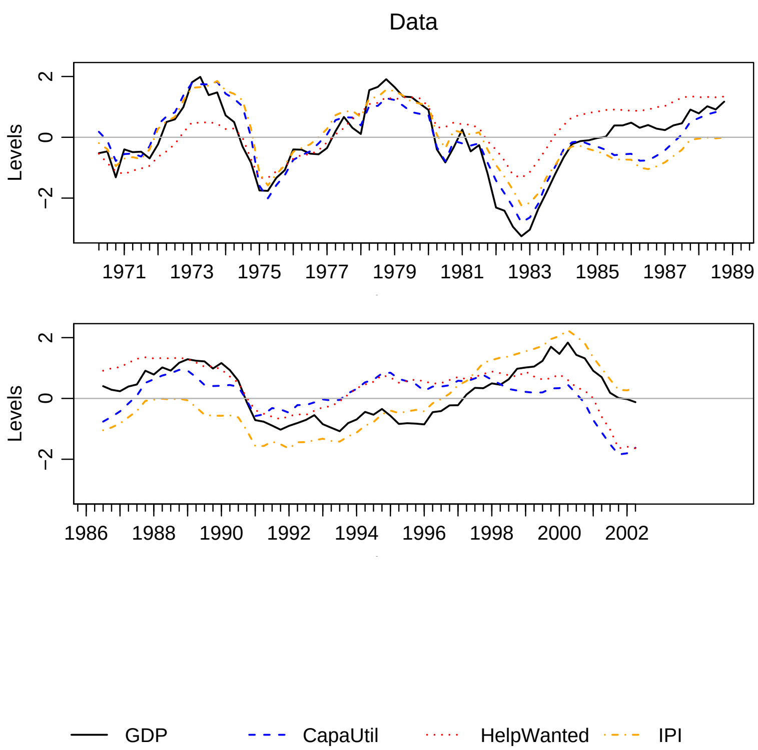

Figure 2: Transformed data

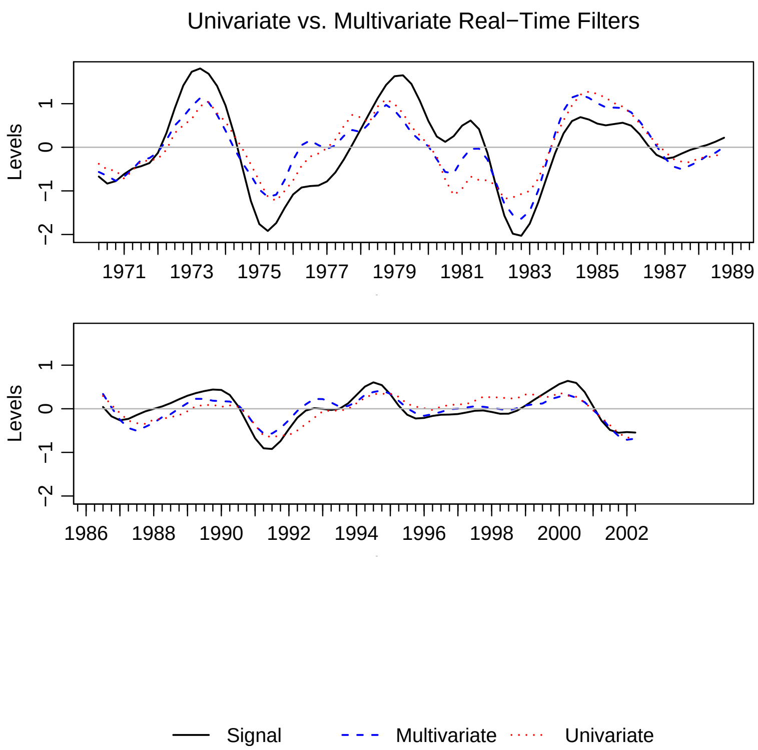

Figure 3: Real-Time Univariate and Multivariate Level-Filters

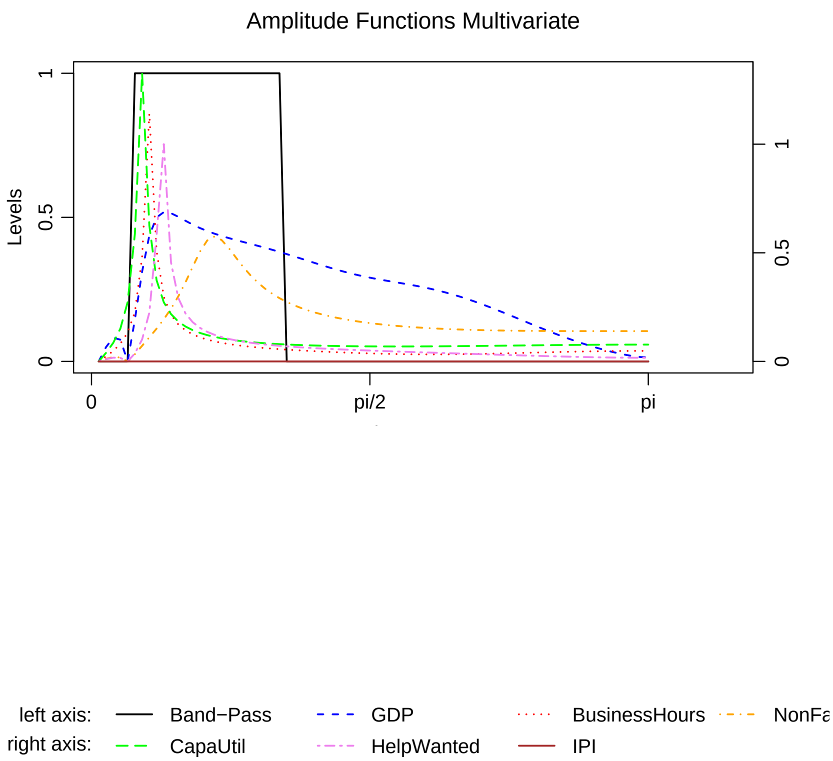

Figure 4: Amplitude Functions Multivariate Level-Filters

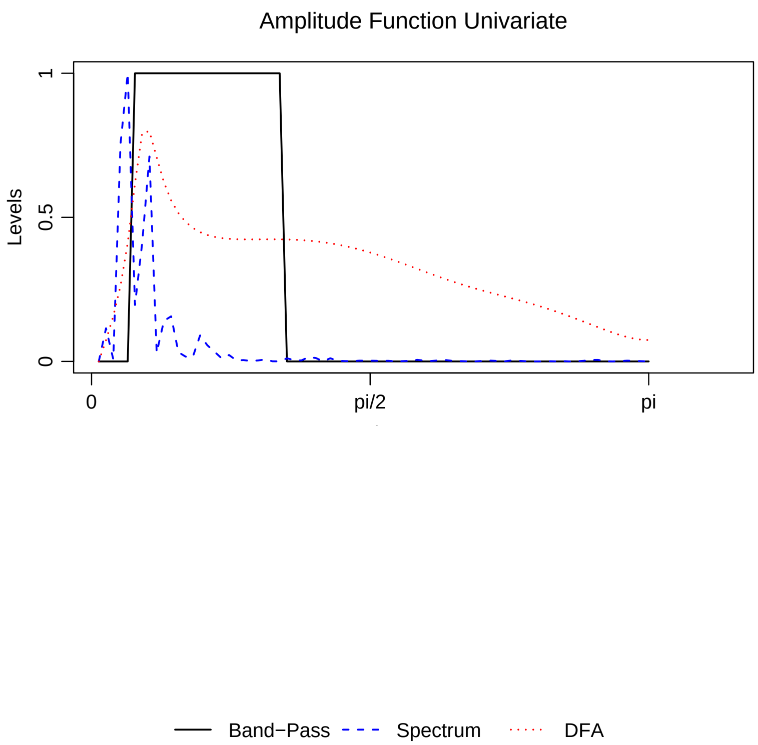

Figure 5: Amplitude Function Univariate Level-Filter

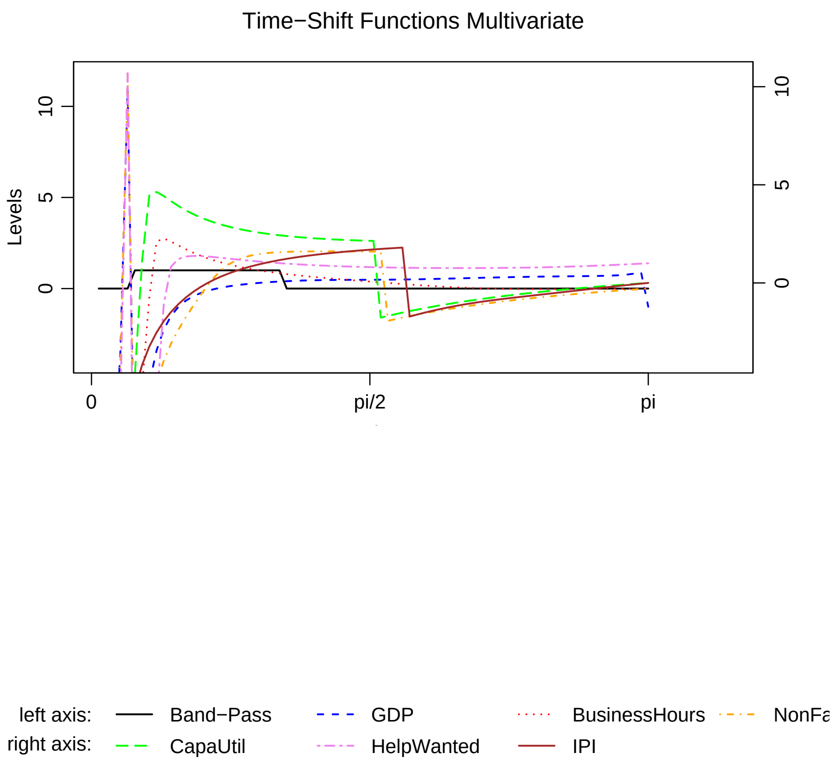

Figure 6: Time-Shift Functions Multivariate Level-Filters

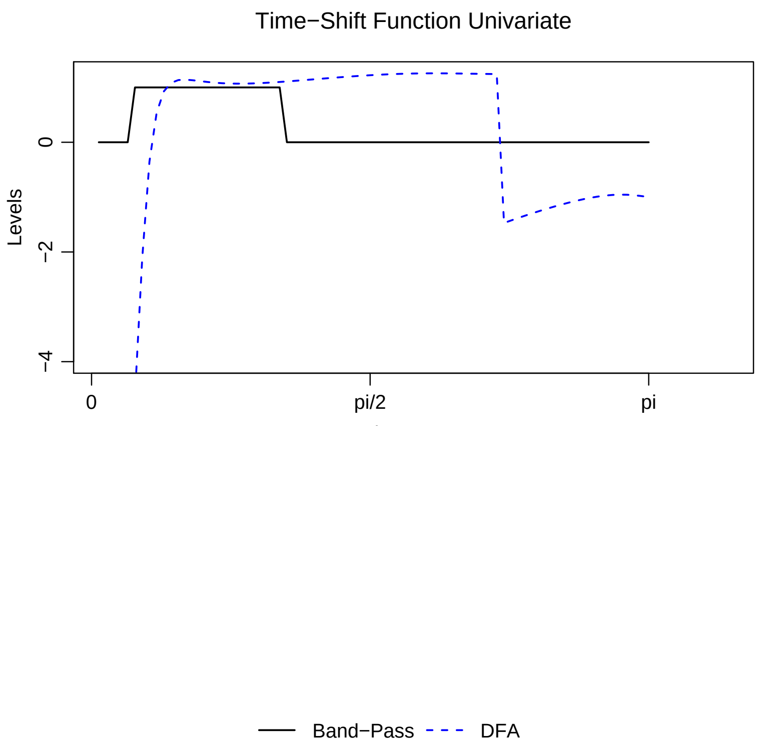

Figure 7: Time-Shift Function Univariate Level-Filter

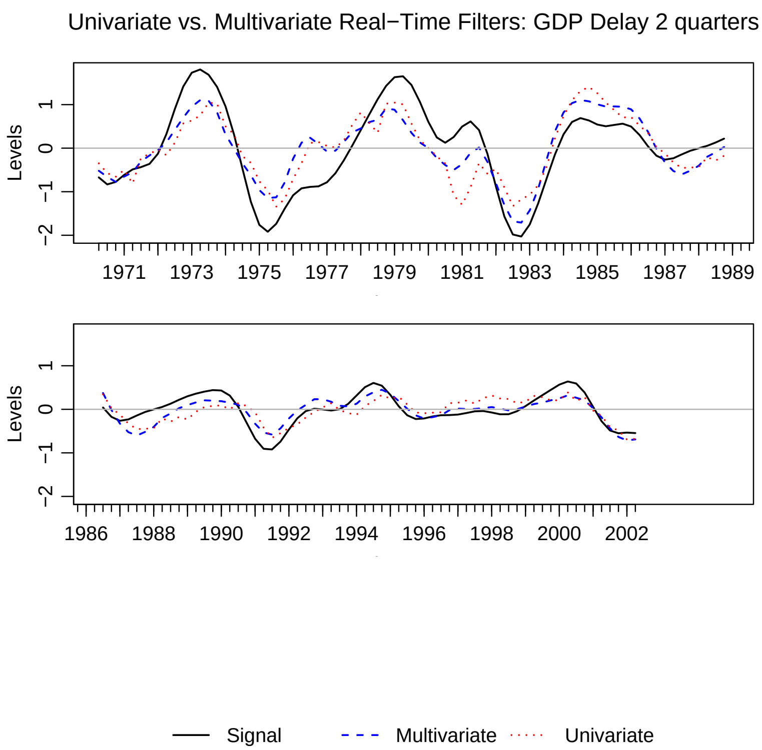

Figure 8: Real-Time Univariate and Multivariate Level-Filters: GDP delayed by 2 quarters

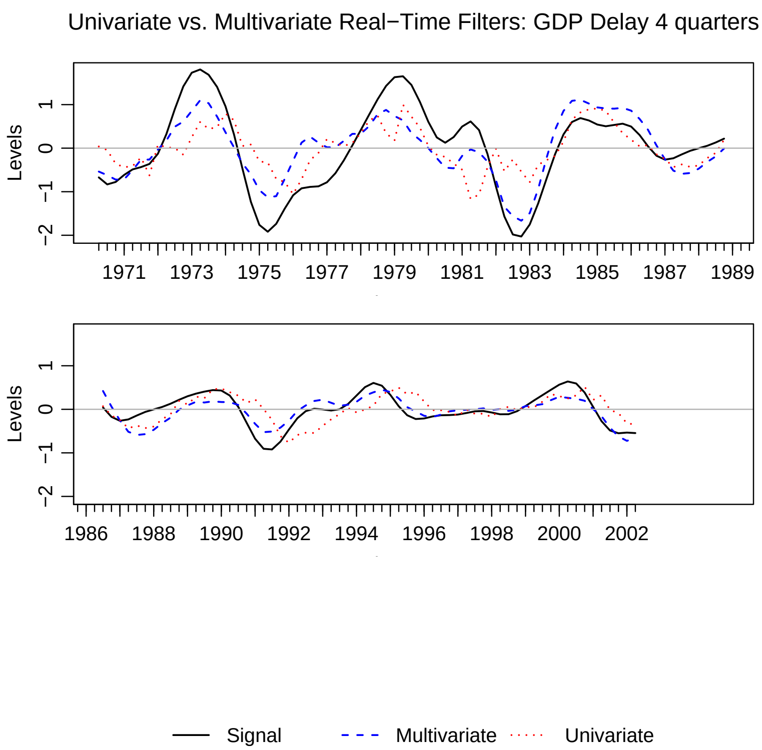

Figure 9: Real-Time Univariate and Multivariate Level-Filters: GDP delayed by 4 quarters

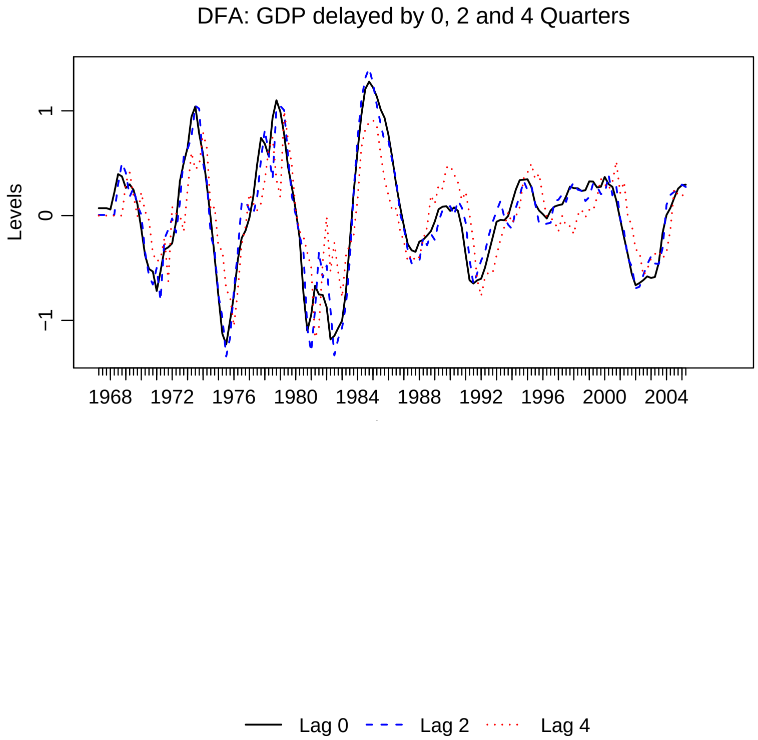

Figure 10: Real-Time Univariate Filters: GDP delayed by 0,2 and 4 quarters

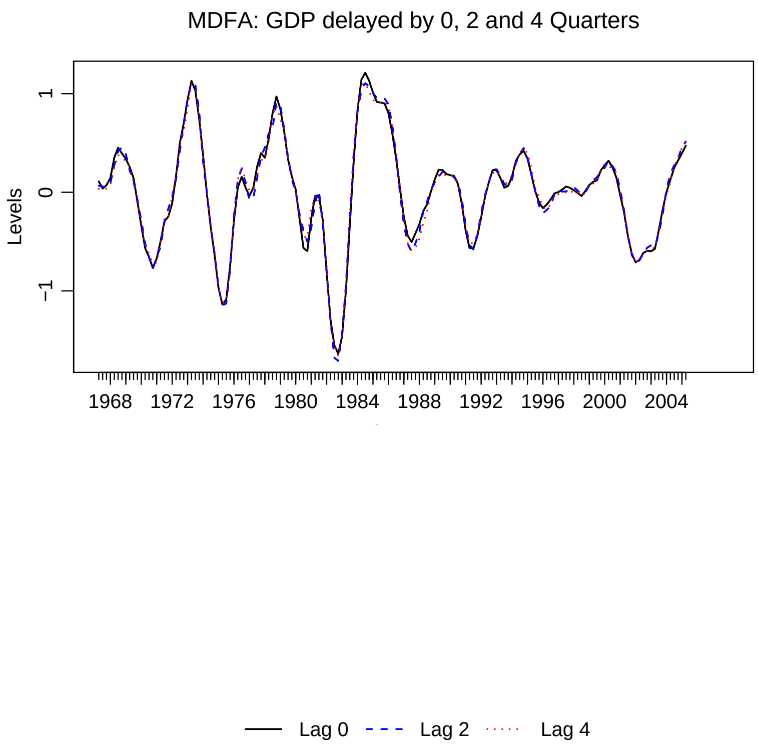

Figure 11: Real-Time Multivariate Filters: GDP delayed by 0,2 and 4 quarters

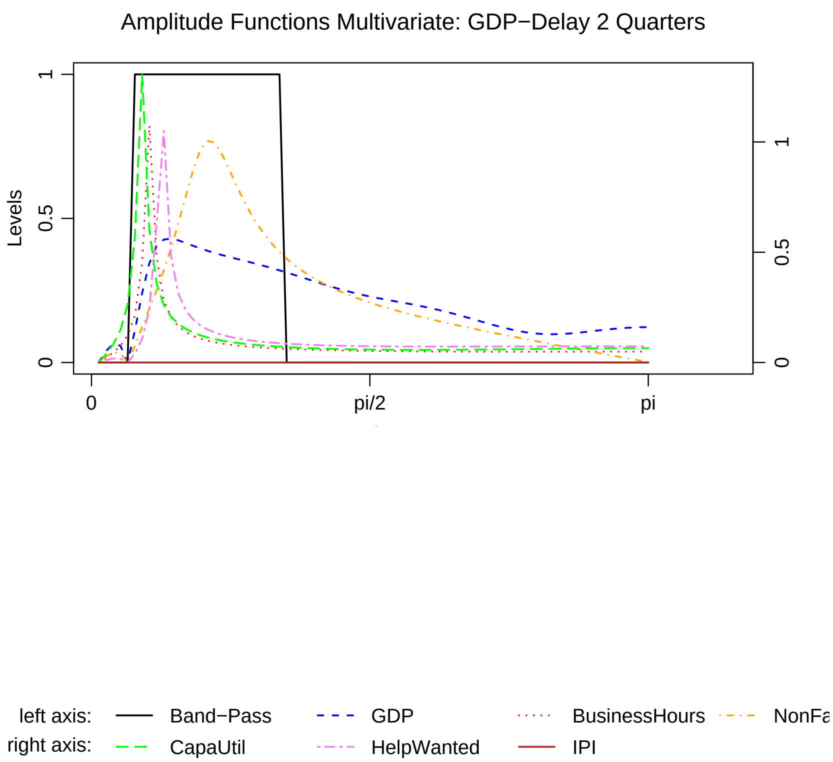

Figure 12: Amplitude Functions Multivariate Level-Filters: GDP lagged by 2 quarters

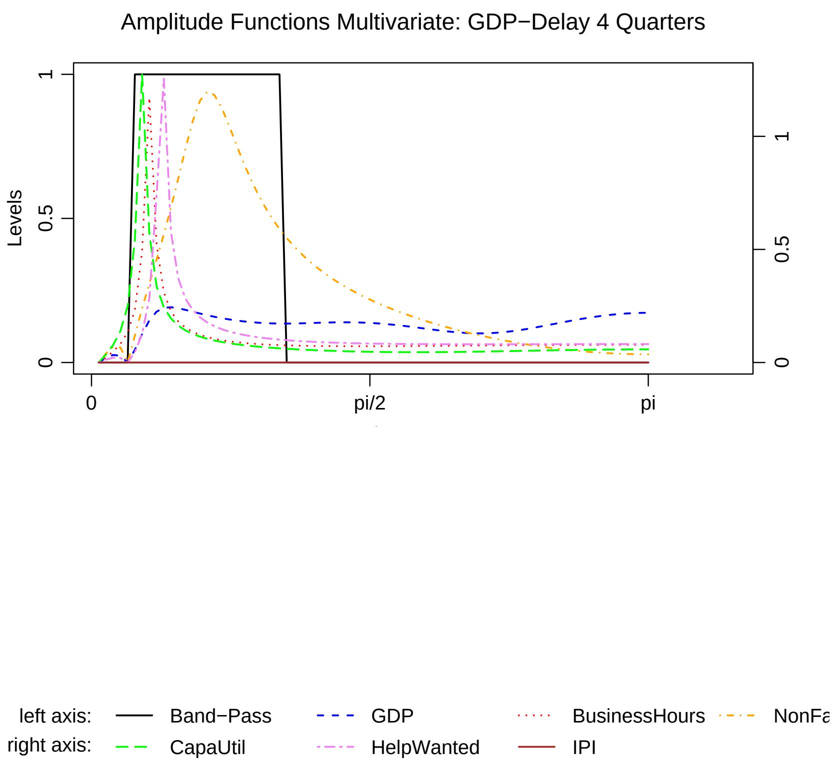

Figure 13: Amplitude Functions Multivariate Level-Filters: GDP lagged by 4 quarters

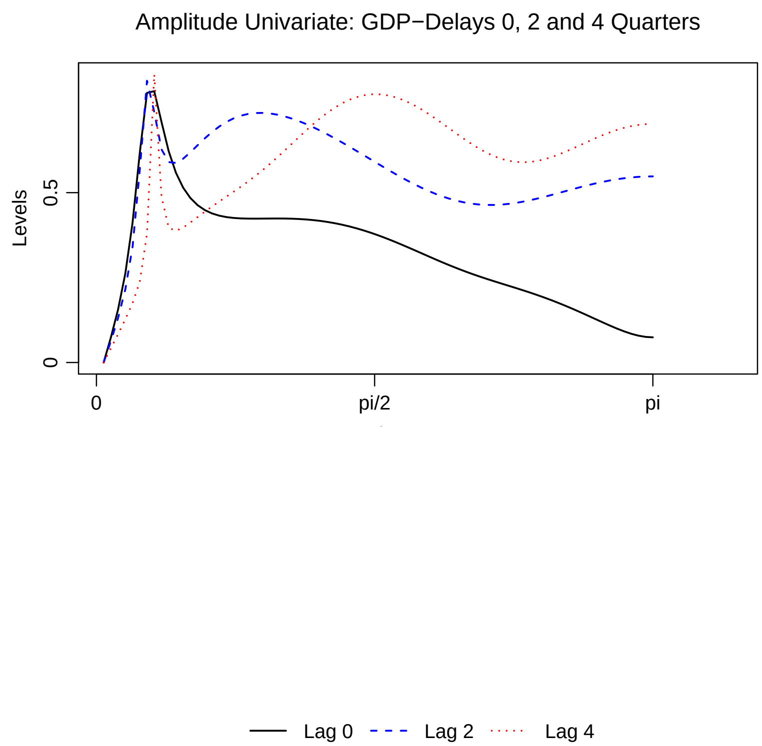

Figure 14: Amplitude Function Univariate Level-Filter: GDP lagged by 0, 2 and 4 quarters

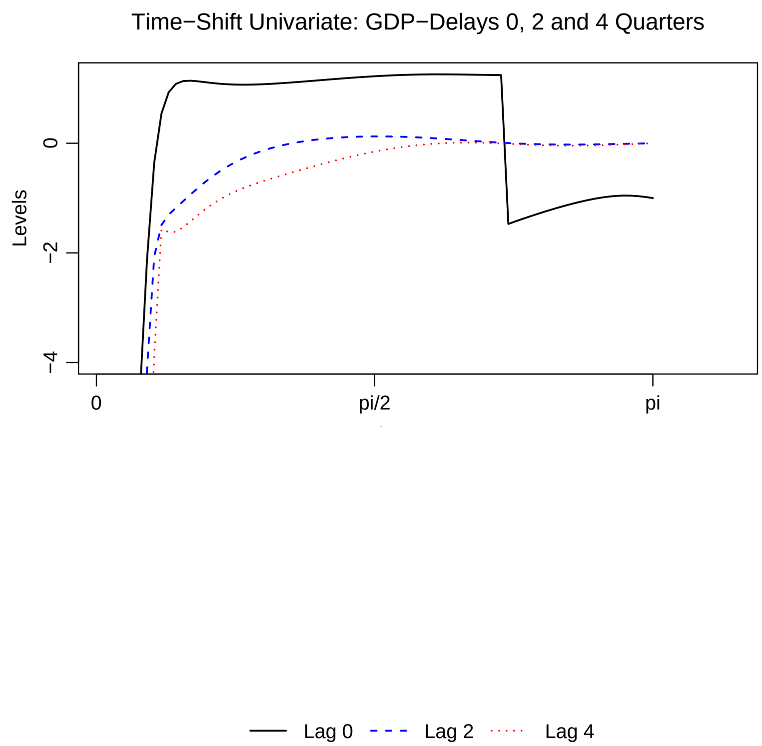

Figure 15: Time-Shift Functions Univariate Level-Filters: GDP delayed by 0, 2 and 4 quarters

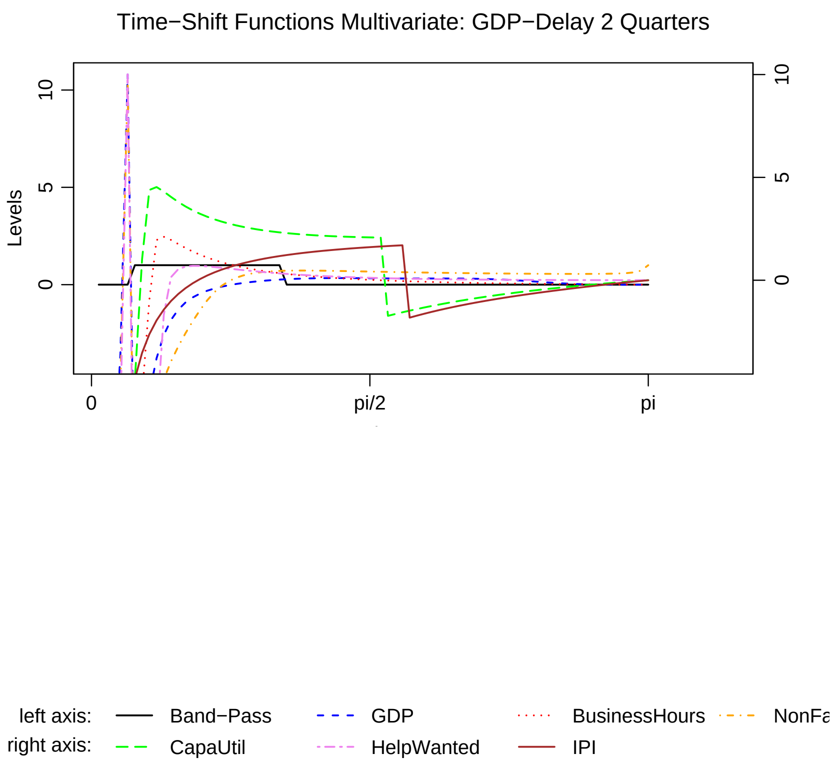

Figure 16: Time-Shift Functions Multivariate Level-Filters: GDP delayed by 2 quarters

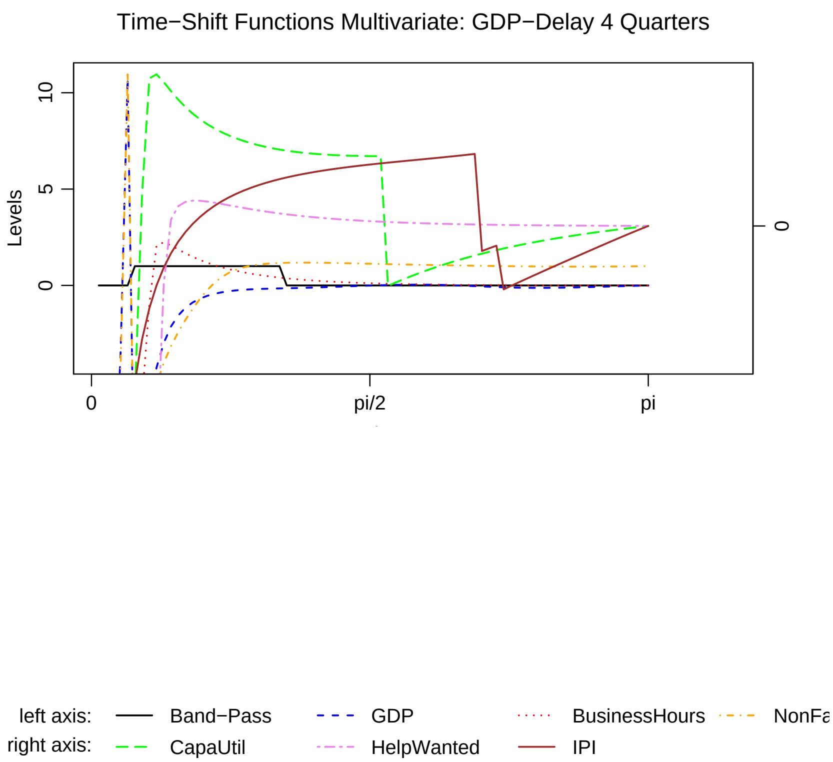

Figure 17: Time-Shift Functions Multivariate Level-Filters: GDP delayed by 4 quarters